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1 author:

Peder Albert Tyvand
Norwegian University of Life Sciences (NMBU)

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A diffusive Boussinesq plume

Peder A. Tyvand

Department of Mathematical Sciences and Technology,
Norwegian University of Life Sciences,
P.O. Box 5003, 1432 Ås, Norway; E-mail: peder.tyvand@umb.no

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Abstract—An axisymmetric Boussinesq plume in a uniform ambient fluid is considered. The classical similarity solution of *Morton et al.* (1956) is modified to account for diffusive losses of buoyancy flux and momentum flux. This leads to a buoyancy flux and a momentum flux that both tend to zero at infinite height. The mass flux at infinity will tend to a finite value that depends on the diffusion parameters for buoyancy and momentum.

Key-words: buoyancy, Boussinesq, diffusion, heat source, plume

1. Introduction

Plumes are relatively slender vertical flows rising above concentrated buoyancy sources. Fire disasters produce strong buoyant plumes. Plumes are important in local meteorology, especially in connection with formation of cumulus clouds. Some basic mathematical solutions for plumes can be found when a singular heat source is taken as the driving mechanism. The most important of these is the celebrated similarity solution of *Morton et al.* (1956), hereafter called the MTT solution.

The MTT solution with the Boussinesq approximation is mathematically compact. It accounts for mass balance, and it preserves momentum and energy within the plume. However, its obvious shortcoming is that it possesses no physical length scale. Any similarity solution will assume that the plume rises to infinite height. According to a similarity solution, the plume velocity will ultimately tend to zero, but the upward mass flux will increase indefinitely. In spite of its mathematical convenience, a similarity solution lacks physical consistency in that it implies an indefinite amount of mass entrainment, up to infinite height. In reality, the entrainment of momentum from outside into the plume must vanish asymptotically at great heights, when the plume velocity slows down sufficiently. Opposing the entrainment, there will be diffusive losses of momentum and buoyancy, from the plume to the surrounding fluid. These diffusive losses introduce physical length scales in the vertical direction. A well-known example of such a length scale is the altitude of a cumulus cloud above the heated ground.

Convective flows in the atmosphere or ocean occur as recirculating convection cells. This means that a plume cannot be considered as an isolated phenomenon. All plumes in the atmosphere and ocean occur as parts of convection cells. This fact suggests another physical shortcoming of the MTT solution. It disregards the fact that a rising plume occupies only a narrow section of the convection cell that it belongs to. A narrow upwelling plume should therefore be surrounded by a broad and slow downwelling flow. However, in the course of this study it becomes clear that the present modifications of the MTT model are insufficient to construct a recirculating convection cell.

The MTT paper succeeded a seminal paper on plumes by *Batchelor* (1954), who also developed similarity solutions. After MTT a lot of papers have followed. To our knowledge, the physical validity of similarity solutions has not been addressed. A recent work by *Scase et al.* (2006) generalizes the axisymmetric MTT model in a fruitful way by starting from the basic hydrodynamic equations to take time-dependence into account.

2. The MTT model

We consider a rising axisymmetric plume in a fluid that would otherwise be at rest. The flow is generated by a singular heat source. The gravitational acceleration is denoted by g. The axisymmetric flow depends on the vertical coordinate z and the radial coordinate r. We define $z = 0$ by the concentrated heat source that drives the flow. We define the radial coordinate r as the horizontal distance from the vertical line through the heat source.

The classical MTT plume model assumes 'top-hat' profiles for the density $\rho(r,z)$

$$\begin{aligned} &\rho(r,z) = \rho(z), \quad && r \le b(z), \\ &\rho(r,z) = \rho_\infty, \quad && r > b(z), \end{aligned} \tag{1}$$

and the vertical velocity $w(r,z)$

$$\begin{aligned} &w(r,z) = w(z), \quad && r \le b(z), \\ &w(r,z) = 0, \quad && r > b(z). \end{aligned} \tag{2}$$

Here we have introduced the plume radius $b(z)$ and the density of the ambient fluid ρ_∞ We will consider only the simplest case, where ρ_∞ is taken constant so that the ambient fluid is assumed uniform. We take the Boussinesq approximation where density variation is included only in the buoyancy term of the momentum equation. The ‘top-hat’ description, Eqs. (1)–(2), assumes that the plume radius $b(z)$ can be sharply defined at each height z. Moreover, it replaces the density and velocity fields within the plume by their average values taken over the plume cross section at each given height z. In the MTT solution, the only communication with the fluid outside the plume is the entrainment of fluid by turbulent mixing into the plume. It is assumed that no loss of momentum or energy from the plume to the surrounding fluid will take place.

The entrainment constant α is introduced by the standard entrainment assumption

$$u_r \Big|_{r=b(z)} = \alpha w, \tag{3}$$

where u_r is the inward radial velocity of the surrounding fluid at the boundary of the plume. This radial entrainment velocity u_r is thus assumed to be proportional to the vertical velocity in the plume at each vertical level z. In this theory, the radial velocity is significant only at the plume boundary. Once the entrained fluid has entered the plume, it is assumed to be thoroughly mixed so that the net average flow becomes vertical. In this averaging procedure one cannot include continuity in radial velocity across the plume boundary.

We introduce the mass flux πQ, the momentum flux πM, and the buoyancy flux πF for the steady plume. By definition we have

$$Q = b(z)^2 w(z) \rho(z), \tag{4}$$

$$M = b(z)^2 w(z)^2 \rho(z), \tag{5}$$

$$F = b(z)^2 w(z) g(\rho_\infty - \rho(z)). \tag{6}$$

This implies the relationships

$$w = M/Q, \;\; \rho = \rho_\infty gQ/(gQ + F), \;\; b = Q/\sqrt{M\rho}, \;\; g' = g(\rho_\infty - \rho)/\rho = F/Q. \tag{7}$$

Here we have introduced the ‘reduced gravity’ g'. The conservation of mass, momentum, and energy in a steady plume is expressed by the equations

$$dQ/dz = 2\alpha\sqrt{\rho_\infty M}, \tag{8}$$

$$dM/dz = QF/M, \tag{9}$$

$$F = F_0 = constant, \tag{10}$$

respectively.

3. Diffusive losses of buoyancy and momentum

Since the buoyancy flux F_0 is constant in the MTT model described above, it parameterizes the singular hot spot in the origin. The energy equation is simply F = constant for a steady plume in homogeneous ambient fluid. Let us take into account a turbulent diffusive loss of energy by postulating the relationship

$$F(z) = F_0 \exp(-\gamma z), \tag{11}$$

where F_0 remains the buoyancy flux specified by the MTT solution, and a spatial decay parameter γ is introduced, assumed to be constant. With this modification, F_0 still parameterizes the hot spot in the origin. The energy equation that is implicitly assumed by the solution, Eq. (11) is

$$dF / dz = -\gamma F , \tag{12}$$

replacing the previous energy conservation equation $dF/dz=0$. The solution, Eq. (11) is a reasonable starting point, because it introduces a physical length scale $1/\gamma$. Since the MTT solution does not contain any length scale, γz immediately constitutes itself as a dimensionless vertical coordinate, which establishes a physical length scale.

A physically consistent description of a diffusive plume must also take into account a diffusive loss of momentum. We modify the momentum equation, Eq. (9) as follows

$$dM / dz = QF / M - \Gamma M , \tag{13}$$

where an additional spatial decay parameter Γ for momentum is introduced, assumed to be constant. Γ is introduced in analogy with the buoyancy decay parameter Γ introduced above. This is a simpler model of turbulent momentum loss than the averaged Navier-Stokes momentum equation with eddy viscosity. However, a Navier-Stokes equation cannot be formulated for the plume in the 'top-hat' description, since the relevant velocity gradients have already been eliminated by the averaging procedure.

We will now derive the plume solution following from the starting point of Eqs. (11) and (13). We still assume that mass is conserved within the plume, with the application of the entrainment hypothesis. We introduce dimensionless length, velocity, mass flux, momentum flux, and buoyancy flux, respectively, by the definitions

$$\hat{z} = \gamma z , \quad \hat{w} = w(\rho_\infty /(\gamma F_0))^{1/3}, \quad \hat{Q} = (\gamma^{5/3} Q)/(F_0^{1/3} \rho_\infty^{2/3}),$$
$$\hat{M} = (\gamma^{4/3} M)/(F_0^{2/3} \rho_\infty^{1/3}), \; \hat{F} = F / F_0 .$$

From now on, we work with dimensionless quantities and drop the hat superscripts. From Eqs. (8) and (13) the governing equations are

$$dQ/dz = 2\alpha\sqrt{M}\,, \qquad dM/dz = QF/M - \beta M\,, \tag{14}$$

expressing conservation of mass and a diffusive loss of momentum. Here we have introduced a dimensionless parameter β defined as

$$\beta = \Gamma/\gamma\,, \tag{15}$$

which may be considered as a turbulent Prandtl number, expressing the relative rate of momentum diffusion compared with buoyancy diffusion. For strong turbulence (large Reynolds numbers), it is plausible that β will be of order unity.

The postulated loss of buoyancy flux is given by

$$F(z) = \exp(-z)\,. \tag{16}$$

In order to solve this set of governing equations we define

$$Q(z) = Q_S(z)\varphi(z)\,, \qquad M(z) = M_S(z)\mu(z)\,, \tag{17}$$

thereby introducing two unknown functions $\varphi(z)$ and $\mu(z)$ that represent the local relative deviations from the steady MTT solution. The MTT similarity solution is represented by $Q_S(z)$ and $M_S(z)$, given by

$$Q_S(z) = (6\alpha/5)(9\alpha/10)^{1/3} z^{5/3}, \;\; M_S(z) = (9\alpha/10)^{2/3} z^{4/3}\,. \tag{18}$$

This is called a similarity solution since it assumes no other length scale than the vertical coordinate itself.

We will now compute the unknown functions $\varphi(z)$ and $\mu(z)$ in this diffusive plume problem. Their boundary conditions are simply

$$\varphi(0) = \mu(0) = 1, \tag{19}$$

since the solution coincides with the MTT solution near the source. The governing equations for $\varphi(z)$ and $\mu(z)$ are determined by inserting their definitions, Eq. (17) into Eq. (14). The resulting equations are

$$(3/5)z\varphi'(z) + \varphi(z) = \sqrt{\mu(z)}\,, \tag{20}$$

$$(3/4)z\mu'(z) + (1 + (3/4)\beta z)\mu(z) = \exp(-z)\varphi(z)/\mu(z)\,. \tag{21}$$

Eqs. (20) and (21) valid for all $z>0$ with the spatial ‘initial’ conditions, Eq. (19). It is worth noting that Eqs. (20)–(21) are independent of the entrainment constant α. The mathematical problem is thus a one-parameter problem in terms of the dimensionless momentum diffusion parameter β. The dependence of buoyancy diffusion is implicit through the definition of dimensionless variables.

This nonlinear system of two first-order Eqs. (20)–(21) will be solved numerically by MATHEMATICA. Because of the factor z accompanying the derivatives, we have to start the integration with a value of z slightly greater than 0. Some results are shown in *Fig. 1* and *2*. *Fig. 1* shows the functions $\varphi(z)$ and $\mu(z)$. *Fig. 2* gives the mass flux $Q(z)$ (upper graphs) and the momentum flux $M(z)$ (lower graphs). The case β=1 is represented by solid curves. The dotted curves represent β=1/3, while the dashed curves represent β=1/*3*. Since β is a kind of turbulent Prandt number, it should be of order unity.

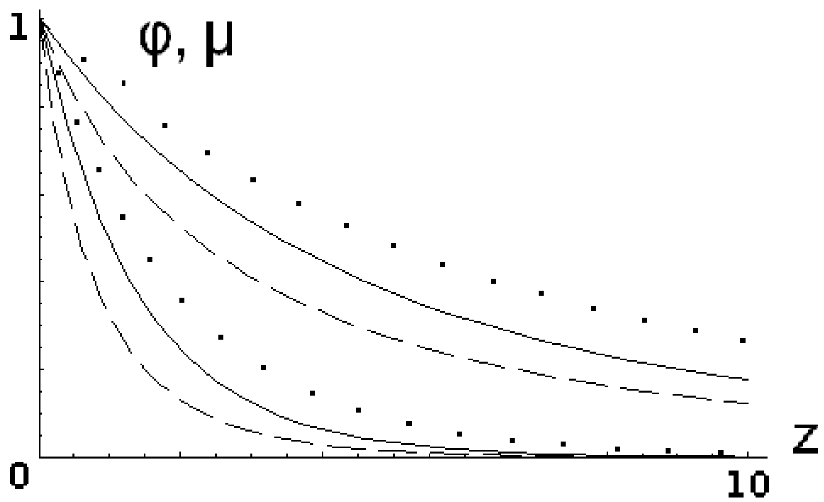

Fig. 1. The functions $\varphi(z)$ (upper curves) and $\mu(z)$ (lower curves). These functions are displayed for β=1/3 (dotted curves), β=1 (solid curves), and β=3 (dashed curves).

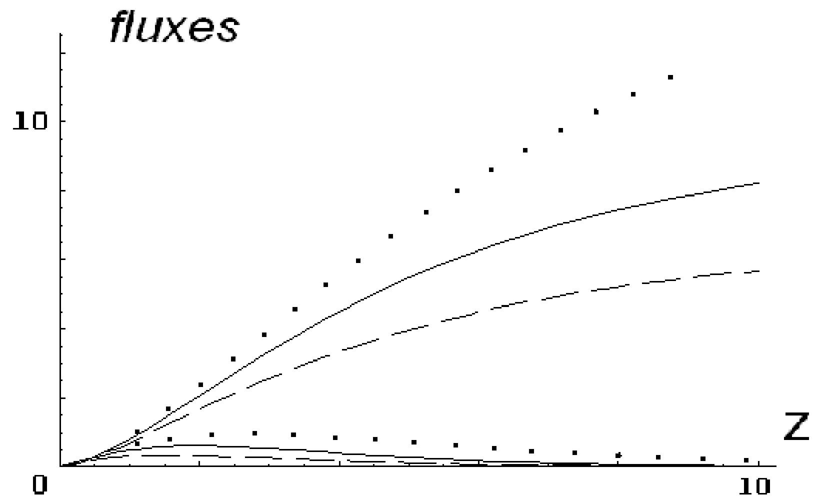

Fig. 2. The upper set of curves represent mass flux $(5/(6\ \alpha))(10/(9\ \alpha))^{1/3}Q(z)$. The lower set of curves represent momentum flux $(10/(9\ \alpha))^{2/3}M(z)$. The functions are displayed for β=1/3 (dotted curves), β=1 (solid curves), and β=3 (dashed curves).

From *Fig. 2* we see that the momentum flux $M(z)$ has a maximum value for a certain value of z, depending on β. On the other hand, there is no maximum value for the mass flux $Q(z)$. Further computations show that it increases with increasing z, and it reaches a constant asymptotic value, dependent on β when $z\to\infty$. In *Table 1* we show $Q(\infty)$ for various values of β, together with the maximal values for $M(z)$. There may be a certain amount of roundoff errors in this system where boundary conditions are specified only at $z=0$, and the numerical integrations for determining $Q(\infty)$ in *Table 1* have all been terminated at $z=50$. The present coupled system of first-order equations seems to defy analytical treatment, since the variables are not separable. *Table 1* shows that $z>1$ at the point of maximum momentum flux, even when the momentum diffusion is relatively strong. This is because the diffusive loss of momentum is compensated by the buoyancy source up to unit height above the heat source.

Table 1. The mass flux at infinite height and the maximum point for the momentum flux for various values of the dimensionless momentum diffusion parameter β

$\boldsymbol{\beta}$	$\mathbf{(5/(6\alpha))(10/(9\,\alpha))^{1/3}Q(\infty)}$	$\mathbf{(10/(9\,\alpha))^{2/3}M_{max}}$	$\boldsymbol{z}$ **at** $\boldsymbol{M=M_{max}}$
0.2	25.02	1.1237	3.141
0.5	12.77	0.8352	2.386
1	9.39	0.6188	1.926
2	7.34	0.4322	1.608
5	5.42	0.2523	1.356

The dimensionless expressions for the plume velocity and plume radius are

$$w(z)=(5/(6\alpha))(9\alpha/(10z))^{1/3}\mu(z)/\varphi(z), \tag{22}$$

$$b(z)=(6\alpha z/5)\varphi(z)/\sqrt{\mu(z)}. \tag{23}$$

In *Fig. 3* we show the radius of the plume as a function of the height, represented by $(5/(6\,\alpha))\,b(z)$, for some values of β. For comparison, the MTT solution is represented by an exact cone that gives the common tangent for these three curves at the apex in the origin.

Contrary to the motivation for this work, it proves impossible to construct an outer solution that gives a closed convection cell, since the mass flux does not tend to zero as $z\to\infty$. There is no balance between sources and sinks in the outer field, so the streamlines will not be closed curves.

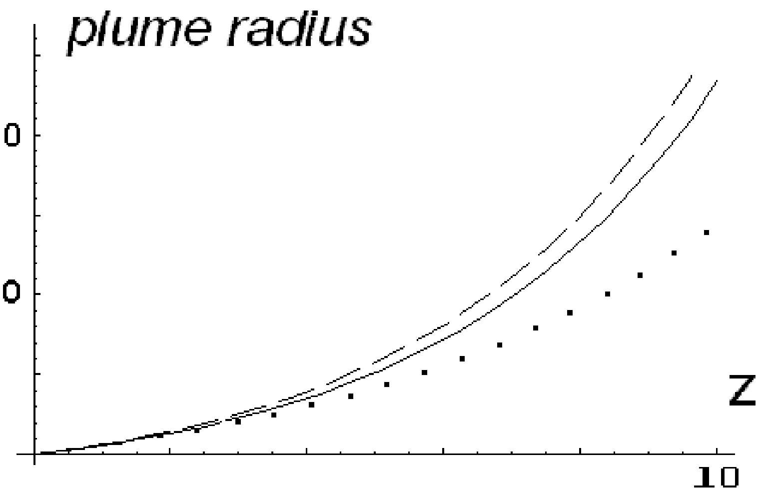

Fig. 3. The plume radius as a function of height represented by $(5/(6\ \alpha))b(z)$. It is displayed for $\beta=1/3$ (dotted curve), $\beta=1$ (solid curve), and $\beta=3$ (dashed curve).

4. Conclusions

An aim of the present work was to model consistently the inner and outer flow field of a plume. In order to achieve this, we included diffusive losses of momentum flux and buoyancy flux from the plume to the ambient fluid, which is assumed of constant density. In order to model the outer flow, it is necessary that the mass flux of the plume ultimately tends to zero with increasing height. The present work shows that this is impossible when the density of the ambient fluid is assumed to be constant. The mass flux in the plume will not tend to zero with increasing height, but it will settle at a constant value. Therefore, no recirculating convection cell can be described by the present type of modeling.

Any model of plume flow in a fluid must be irreversible in time. In the classical MTT solution, the only entropy producing mechanism is the mass entrainment into the plume. As a contrast, the present model takes into account three irreversible phenomena: (i) Turbulent entrainment from the ambient fluid, incorporated into the mass balance. (ii) Turbulent diffusive loss of momentum flux to the ambient fluid. (iii) Turbulent diffusive loss of buoyancy flux to the ambient fluid. While these three phenomena are still being modelled in a highly simplified way, their descriptions in the present work are mutually consistent.

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