Design of fishing exclusion zones

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Outline

Problem statement

The goal is to:

- Ensure the survival of endangered fish by achieving balance between fishing, birth, and movement rates.
- Compare the effectiveness of different geometries of exclusion zones.
- Improve on the current mathematical model by modifying the source term.
- Finding parameter values using known data.

Single species model

Reaction-diffusion equation:

$$\theta_t = -\nabla.\underline{q} + S(\theta, \underline{x}). \tag{1}$$

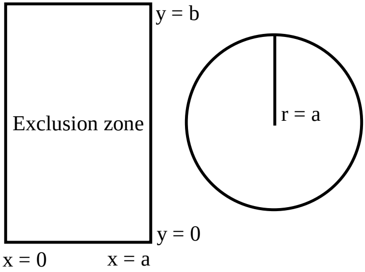
Using Fick's law we get

$$\theta_t = \nabla.\left[D(\theta)\nabla\theta\right] + f(\underline{x})R(\theta). \tag{2}$$

Possible modifications:

- Consider a time-dependent domain. Using the Reynolds transport theorem we can show that equation (1) still holds with $\underline{q} = -D(\theta)\nabla\theta + \theta\underline{u}$ where $\underline{u}$ is the velocity of the domain.
- Previous models for circular geometries use $f(r) = 1$ and $R(\theta) = s\theta(1 - \theta/m)$. We will consider a Gaussian distribution.

Comparison of different geometries



We first linearise the governing equation for small θ and small $|\nabla\theta|$ which gives

$$\theta_t = D(0)\nabla^2\theta + s\theta. \tag{3}$$

Comparison of different geometries

Now consider a rectangular domain with length a and width b. For now let $f(x,y) = 1$. Use separation of variables to get a basis of solutions. We find that

$$\theta = A \exp[A_{lm} t] \sin(xl\pi/a) \sin(ym\pi/b), \tag{4}$$

where

$$A_{lm} = -D_0 \left(\frac{m^2\pi^2}{b^2} + \frac{l^2\pi^2}{a^2} \right) + s. \tag{5}$$

For $A_{11} > 0$, the population will increase. This gives a result in terms of the hyperbolic rms length

$$\frac{1}{\sqrt{1/(2b^2) + 1/(2a^2)}} > 2\pi\sqrt{D(0)/s}. \tag{6}$$

Comparison of different geometries

Rearranging allows us to write this expression as a ratio of area to diagonal

$$\frac{ab}{\sqrt{a^2+b^2}} > \pi\sqrt{2D(0)/s}. \tag{7}$$

For a square we have

$$a > 2\pi\sqrt{D(0)/s} \tag{8}$$

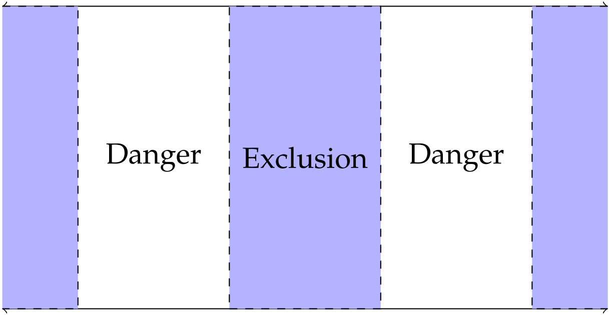
- Reduction of 2-D model to 1-D model gives

$$a > \pi\sqrt{D(0)/s}. \tag{9}$$

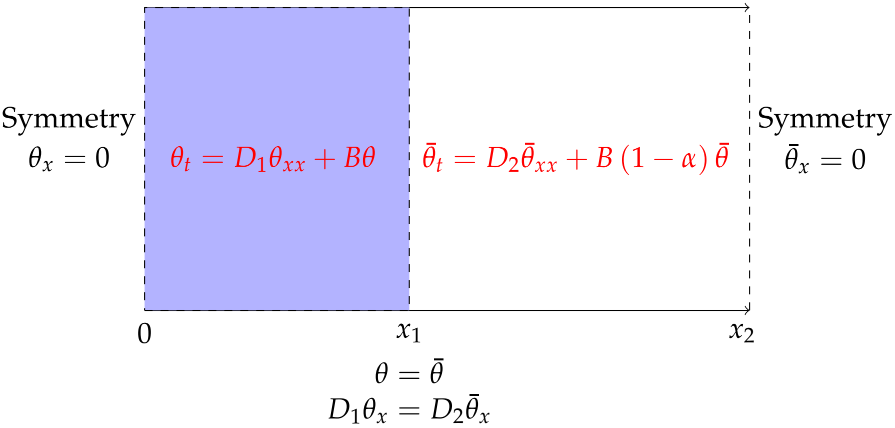
- For a circular geometry

$$a > \lambda_1\sqrt{D(0)/s}, \quad J_0(\lambda_1) = 0. \tag{10}$$

Infinite series of zones



Dimensional model



Dimensionless model

Employ the scaling

$$x = x_1 x', \qquad t = \frac{x_1^2}{D_1} t'$$

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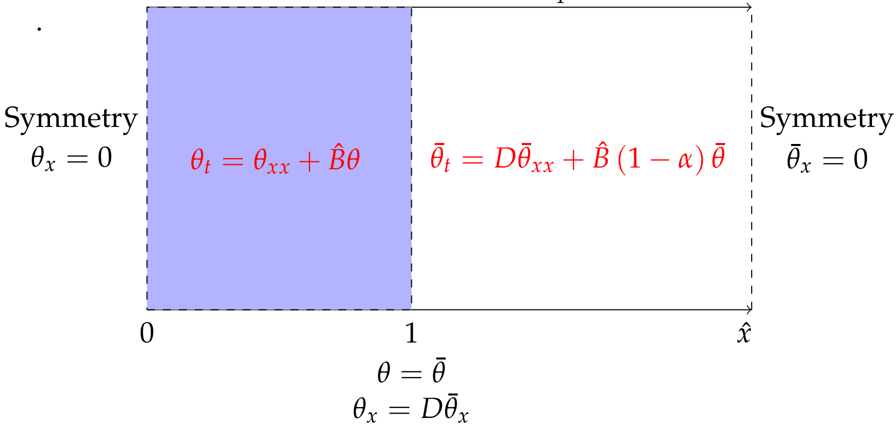

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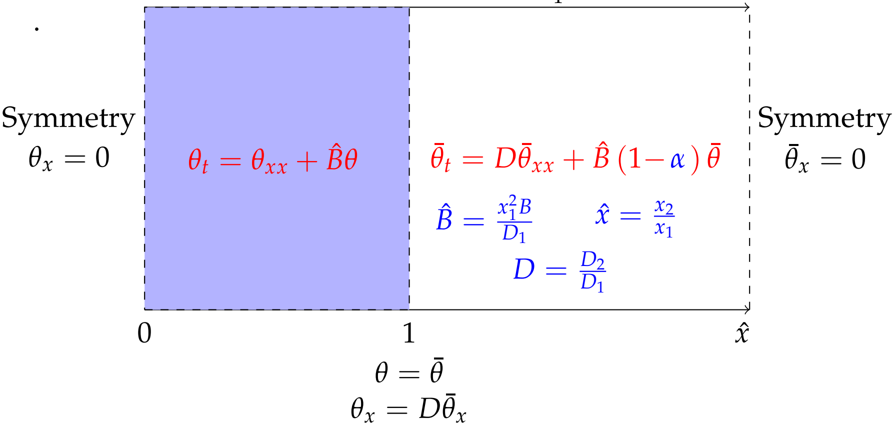

Solution in zone I

- Variable separable solution for the population density function $\theta = X(x)T(t)$ furnishes the general solution in I

$$T = T(0)\exp\left(\lambda^2 t\right) \tag{11}$$

- Two cases

$$\textbf{Case 1}: X(x) = A_1\cos(\omega_1 x), \;\; \hat{B} - \lambda^2 > 0 \tag{12}$$
$$\textbf{Case 2}: X(x) = A_1\cosh(\omega_1 x), \;\; \hat{B} - \lambda^2 < 0 \tag{13}$$

where $\omega_1^2 = Abs[(\hat{B} - \lambda^2)]$.

Solution in zone II

- Variable separable solution for the population density function $\theta = X(x)T(t)$ unearths the general solution in II

$$T = T(0)\exp\left(\lambda^2 t\right) \tag{14}$$

- Two cases, but we are really interested in the case where $\alpha > 1$.

$$\textbf{Case A}: X(x) = A_1\left(\cosh(\overline{\lambda}x) - \tanh(\overline{\lambda}x_2)\sinh(\overline{\lambda}x)\right) \tag{15}$$

where

$$-\overline{\lambda}^2 = \hat{B}[(1-\alpha) - \lambda^2] \tag{16}$$

Interface conditions

The dimensionless interface conditions at $x = x_2$ are

$$\theta = \overline{\theta}, \quad \theta_x = D\overline{\theta_x}. \tag{17}$$

For Case 1 and Case A this leads to

$$\frac{\sqrt{\hat{B}-\lambda^2}\coth\left[(\hat{x}-1)\sqrt{\hat{B}-\hat{B}\alpha-\lambda^2}\right]\tan\left[\sqrt{\hat{B}-\lambda^2}\right]}{D\sqrt{\hat{B}-\hat{B}\alpha-\lambda^2}} = 1. \tag{18}$$

For $\lambda = 0$:

$$\frac{\coth\left[(\hat{x}-1)\sqrt{\hat{B}-\hat{B}\alpha}\right]\tan\left[\sqrt{\hat{B}}\right]}{D\sqrt{1-\alpha}} = 1. \tag{19}$$

Condition for marginal stability

- The hyper-surface in $(\hat{B},\ \alpha,\ D,\ \hat{x})$ space defined by

$$\frac{\coth\left[(\hat{x}-1)\sqrt{\hat{B}(1-\alpha)}\right]\tan\left[\sqrt{\hat{B}}\right]}{D\sqrt{1-\alpha}} = 1, \qquad (20)$$

  separates region of different stability.
- Can use (20) to evaluate conservation strategies.

Conservation strategy

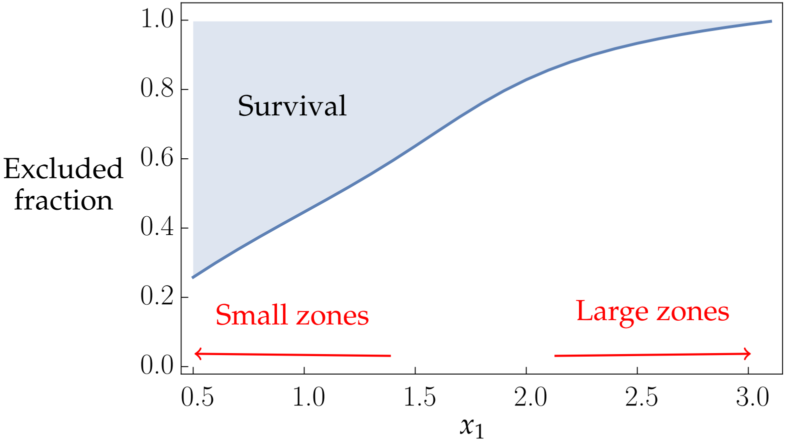

Figure: Fraction of river that must be excluded. Sample parameters: $D = 1$, $\alpha = 5$, and $B = 1$

Stability analysis for modified reactive diffusion model

- Environmental heterogeneous factor is a parabolic curve.
- Model

$$\theta_t = D\theta_{xx} + \left(1 - \epsilon x^2\right) B\theta \tag{21}$$

- Variable separable solution $\theta = e^{At}Q(x)$, where

$$Q(x) = e^{-\frac{1}{4}x^2} M\left(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}x^2\right) \tag{22}$$

and

$$a = \frac{1}{2}\frac{A - B}{D(0)^{\frac{1}{2}}}\left(B\epsilon\right)^{-\frac{1}{2}}$$

Linear stability analysis

$$\theta_t = D(0)\theta_{xx} + s\theta f(x),$$

$$\theta = P(t)Q(x),$$

$$A = \frac{P'(t)}{p} = D(0)\frac{Q''(x)}{Q} + s(1 - \epsilon x^2)$$

$$Q'' + \left[\frac{s - A}{D(0)} - \frac{\epsilon s x^2}{D(0)}\right] Q = 0,$$

$$\overline{x} = \frac{x}{\ell}, \ell = \frac{1}{\sqrt{2}}\left[\frac{D(0)}{\epsilon s}\right]^{\frac{1}{4}}.$$

$$Q'' - \left[a + \frac{1}{4}\overline{x}^2\right] Q = 0,$$

$$a = \frac{1}{2}\left[\frac{D(0)}{\epsilon s}\right]^{\frac{1}{2}} \frac{A - s}{D(0)}.$$

First approximation linear stability analysis

We consider the even solution

$$Q(x) = l^{-2} e^{-\frac{1}{4}\overline{x}^2} M\left(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\overline{x}^2\right),$$

First zero approximation (Abramowitz Stegun) occurs when

$$\begin{aligned} \overline{x}^2 &= \frac{\pi^2 \left(1 + \frac{1}{4} - \frac{3}{4}\right)^2}{1 - 2a - 1}, \\ \bar{x}^2 &= \frac{\pi^2}{-4a}. \end{aligned}$$

Stability cross-over

$$2x_1 > \pi \left[\frac{D(0)}{s}\right]^{\frac{1}{2}},$$

which concurs with findings above. Next correction by the Newton-Raphson method is less than 1% ($f = 1$ at bddry).

Finding parameter values using data

- We have 6000 data points for fish movement.
- Fit the probability density function:

$$P = \alpha(\pi D_1 t)^{-1/2} \exp[-x^2/4D_1 t] + (1-\alpha)(\pi D_2 t)^{-1/2} \exp[-x^2/4D_2 t]. \quad (23)$$

So we have two sub-populations, namely, home-bodies and travellers which was considered in the previous model.

Statistical confirmation

- Data maintained on Oceanographic Research Institute's Cooperative Fish Tagging Project.
- Need to confirm model with paper by Bruce et al 2016.
- The diffusivity coefficient $D = \frac{{x_i}^2}{2\pi t_i}$ will be furnished by random walk analysis.

Numerical simulations

- Lie point symmetries of the non-linear equations.
- Perturbation methods.
- Numerical solutions of the full non-linear equations.
- Question the applicability of Fick's law.
- Consider a moving boundary.

Conclusions

- The square is the best geometry!
- A simple 1-D model provides a simple framework for assessing conservation strategies.
- Including the parabolic adjustment gives no further information at the lowest order.

"So long and thanks for all the fish!"