Turbulent Fluid Fracture

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Overview

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Problem description

Descriptive Equation

$$\frac{\partial h}{\partial t} = -D^* \frac{\partial}{\partial x}[(-h^3 \frac{\partial h}{\partial x})^{\frac{1}{m+2}}] \qquad (1)$$

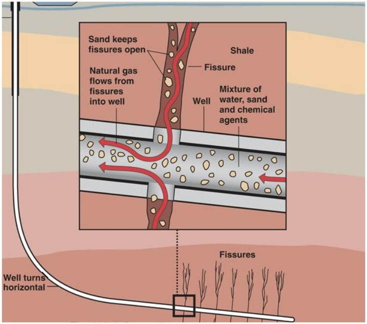

Figure: A

Derivation

$$\frac{2\tau_0}{\rho\bar{u}^2} = n[\frac{2\rho h\bar{u}}{\mu}]^m = nRe^m \qquad \text{where } Re = \frac{2\rho h\bar{u}}{\mu} \tag{2}$$

$$\tau_0 = -h\frac{\partial p}{\partial x} \tag{3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = 0 \tag{4}$$

The use of equations (2) - (4), Assumption: $p(x,t) = \Lambda h(x,t)$

$$\frac{\partial h}{\partial t} = -D^* \frac{\partial}{\partial x}[(-h^3\frac{\partial h}{\partial x})^{\frac{1}{m+2}}] \tag{5}$$

where $D^* = D\Lambda^{\frac{1}{m+2}}$ and $D = \left[\frac{2^{(1-m)}\mu^m\rho^{-(1+m)}}{n}\right]^{\frac{1}{m+2}}$

Analytical solution

Reduction: PDE to ODE

- **Scaling Transformations**

$$\bar{t} = \lambda^a t, \qquad \bar{x} = \lambda^b x, \qquad \bar{h} = \lambda^c h$$

- **Invariance of the PDE**

$$c = \tfrac{m+2}{m-2} a - \tfrac{m+3}{m-2} b$$

This results in Quasi linear equation

$$cf = bx\frac{\partial f}{\partial x} + at\frac{\partial f}{\partial t} \tag{6}$$

- General Solution to the Quasi linear equation

$$h = f(x,t) = t^{\frac{c}{a}} F(\xi) \tag{7}$$

- Transformation $\xi = \frac{x}{t^{\alpha}}$ where $\alpha = \frac{b}{a}$.

- ODE

$$D^* \frac{d}{d\xi}[(-F^3 \frac{dF}{d\xi})^{\frac{1}{m+2}}] - \alpha \frac{d}{d\xi}(\xi F) + \frac{m+2-5\alpha}{m-2} F = 0 \tag{8}$$

Simplification of ODE

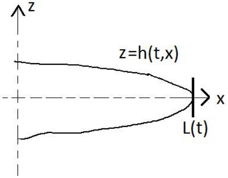

Figure: B

- **moving boundary condition $L(t)$**

First particular solution

- **Elimination of the third term in eqn.:**

  Let $\alpha = \frac{1}{5}(m+2)$

- **Boundary condition**

  $H(1) = 0$

- **Integration of eqn. w.r.t.**η**,**
  Assumption:$(-H^3(1)H'(1))^{\frac{1}{m+2}}$

$$-H^3 \frac{dH}{d\eta} = (\alpha \eta H)^{m+2} \tag{10}$$

- **Integration of eqn.q w.r.t.**η**, Condition:** $m < 2$

$$H = [\alpha^{m+2} \frac{2-m}{m+3} (1 - \eta^{m+3})]^{\frac{1}{2-m}} \tag{11}$$

Comparison of Laminar and Turbulence cases:

Laminar	Turbulence
m=-1	m=$-\frac{1}{4}$
$\alpha = \frac{1}{5}$	$\alpha = \frac{7}{4}$
$L(t) = L_0 t^{\frac{1}{5}}$	$L(t) = L_0 t^{\frac{7}{4}}$

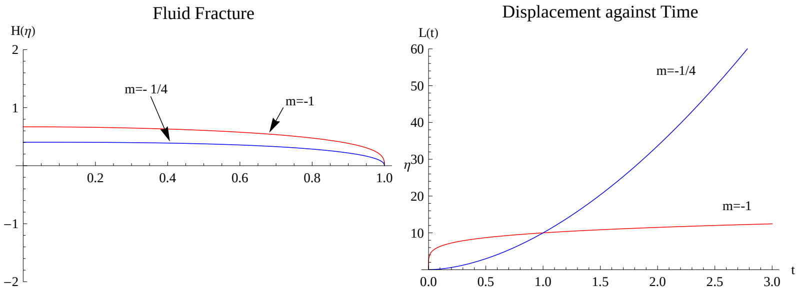

Figure: C

Second Solution

Ansatz:

$$H(\eta) = A(1-\eta)^p \tag{12}$$

$$A = \left[\frac{(2-m)^{\frac{1}{m+2}}(m+2)}{m+4}\right]^{\frac{m+2}{2-m}},$$

$$p = \frac{1}{2-m}, \qquad \alpha = 1.$$

Hence

$$H(\eta) = \left[\frac{(2-m)^{\frac{1}{m+2}}(m+2)}{m+4}\right]^{\frac{m+2}{2-m}}(1-\eta)^{\frac{1}{2-m}} \tag{13}$$

Comparison of Laminar and Turbulence cases:

Laminar	Turbulence
m=-1	m=$-\frac{1}{4}$
$\alpha = 1$	$\alpha = 1$
$L(t) = L_0 t$	$L(t) = L_0 t$

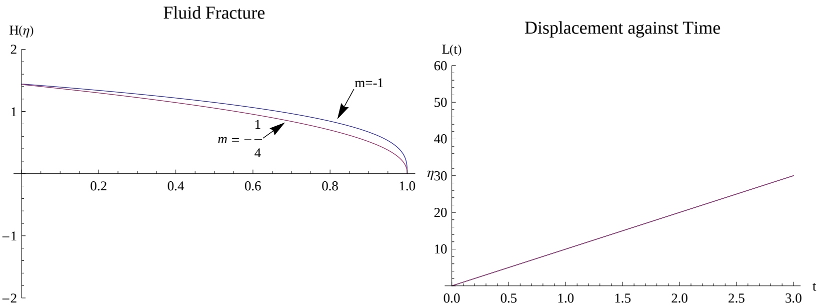

Figure: D

Conclusion

- Turbulence case appears to be fracturing faster than the laminar case.
- Value of m can give us more information about the comparison of the two cases.
- Phase plots could help in anaysis of the stability of the solutions obtained.

Q and A