A MATHEMATICAL MODEL OF A ROAD BLOCK

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Abstract: The effect of a road block on traffic flow is investigated using the model in which the traffic velocity is a linear function of the traffic density. The road consists of two lanes in one direction and one lane is closed for a short distance by a road block. The length of the tailback at the entrance to the road block, the time spent in the road block and the traffic flux at the exit to the road block are calculated. The maximum length of the tailback is a linear function of the time T^* that the road block was in place. The effect of the road block on the traffic depends on three parameters, the density of the oncoming traffic, the time T^* and the ratio λ of the speed limit in the road block to the speed limit on the open road. The congestion caused by the road block can be managed by adjusting T^* and λ.

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1 Introduction

In the fight against crime, the police frequently use road blocks to stop and search cars. A road block usually has two stages; first the traffic is diverted into a single stream and then a fraction of the passing cars are waved down and searched. The modelling of a road block was brought to the Sixth South African Study Group with Industry in 2009 by the Gauteng Department of Community Safety who wanted to know how a search strategy would impact on traffic flow. In this paper we address the first stage of the road block by modelling the effect on traffic flow of the reduction of a two lane road to a single lane with a given speed restriction.

A continuum model for traffic flow ([1] to [5]) is used which assumes that the traffic is travelling steadily along a long road with no side turnings before the road block is set up. A linear relationship between the traffic density and the traffic speed is assumed. If the traffic flow is sufficiently heavy a shock wave will form at the entrance to the road block and it will propagate backwards through the oncoming traffic. When operating the road block it is important to know the length of the tailback and the reduction in the traffic flux due to the road block. At the same time a shock wave forms at the exit to the road block and propagates in the direction of the traffic flow. At time T^* the road block is removed. We will investigate the evolution of the shock waves and the time taken for the effects of the road block to clear.

The results may be generalised to a highway with n lanes reduced to m lanes by a road block where $m < n$. An accident in which one or more lanes of a highway are blocked is another example of a road block. When all the lanes are blocked the effect is the same as that of a red traffic light [2].

An asterisk will be used to distinguish dimensional variables. Variables without an asterisk are dimensionless.

2 Road block in traffic flow

At $t = 0$ a road block, $0 \leq x \leq L$, is introduced in the traffic flow reducing the two lanes of traffic to one lane. The traffic flux in the open road and in the road block are

$$q_L = \rho_L(1 - \rho_L) \ , \quad q_R = \lambda\rho_R(1 - 2\rho_R) \ . \tag{3.1}$$

Graphs of q_L and q_R plotted against ρ are presented in Figure 1. The maximum traffic flux in the open road is $q_{L\,\max} = 1/4$ and occurs for $\rho_L = 1/2$ while the maximum traffic flux in the road block is $q_{R\,\max} = \lambda/8$ and occurs for $\rho_R = 1/4$. When $\rho_R = 1/4$, the velocity of the traffic in the road block is $\lambda/2$ which in dimensional variables is one half the speed limit. Since the maximum traffic flux in the road block is $\lambda/8$ a shock wave will form at the entrance to the road block when the traffic flux in the open road is in the range $\lambda/8 < q_L \leq 1/4$. When $q_L = \lambda/8$, the density ρ_L satisfies the quadratic equation

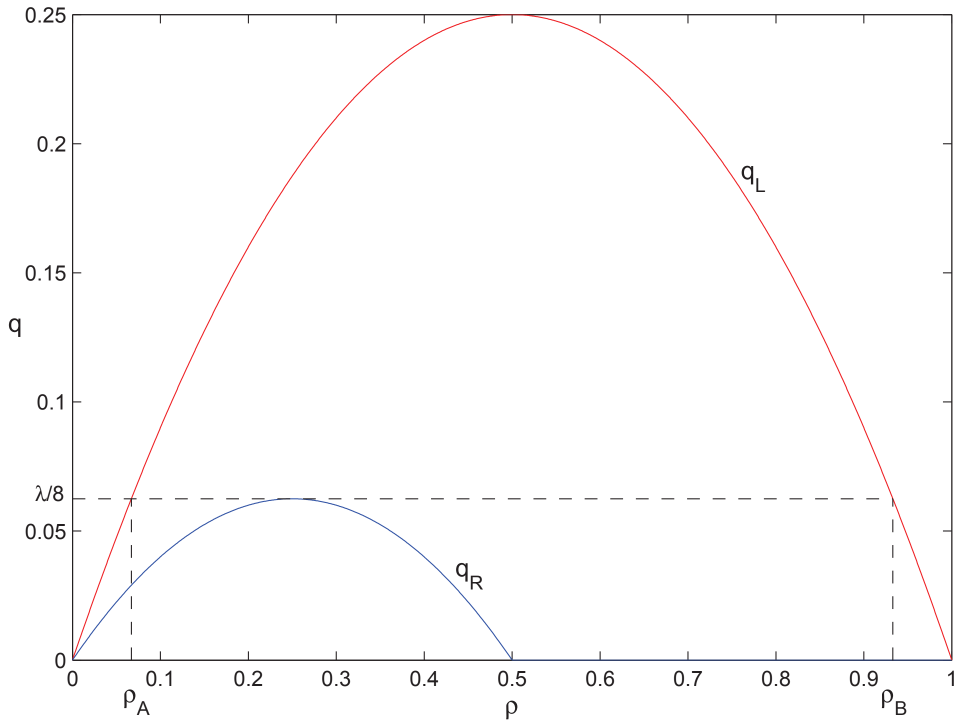

Figure 1: The traffic flux in the open road, q_L, and in the road block, q_R, plotted against the traffic density ρ for $\lambda = 1/2$. The traffic densities ρ_A and ρ_B are defined by (3.3) and (3.4).

3 Mathematical model

Consider an open road, $-\infty < x^* < \infty$, consisting of two lanes in one direction with no off-ramps or on-ramps. The flow of traffic is in the positive x^*-direction. It is reduced to one lane at a road block, $0 < x^* < L^*$.

The traffic density $\rho^*(x^*, t^*)$ is the number of vehicles per unit length of the road. The velocity of a vehicle is denoted by $v^*(x^*, t^*)$. The traffic flux $q^*(x^*, t^*)$ is the number of vehicles passing position x^* per unit time. Denote by ρ_L^*, V_L^* and q_L^* the traffic density, velocity and flux in the open road to the left of the shock wave and by ρ_R^*, V_R^* and q_R^* the corresponding quantities in the road block. Then

$$q_L^* = \rho_L^* V_L^* \ , \qquad q_R^* = \rho_R^* V_R^* \ . \tag{2.1}$$

The maximum traffic density occurs when the cars are bumper-to-bumper. Let $\rho_{L\max}^*$ and $\rho_{R\max}^*$be the maximum traffic density in the open road and in the road block. Since the two lanes are reduced to one lane in the road block,

$$\rho_{R\max}^* = \frac{1}{2}\, \rho_{L\max}^* \ . \tag{2.2}$$

The maximum velocity in the open road and in the road block are the speed limits $V_{L\max}^*$ and $V_{R\max}^*$ where $V_{R\max}^* \leq V_{L\max}^*$. Let

$$\lambda = \frac{V_{R\max}^*}{V_{L\max}^*} \leq 1 \ . \tag{2.3}$$

The lower limit $\lambda = 0$ could describe a red traffic light in which the traffic is brought to a stop for an interval of time or a road block in which all the cars are stopped and checked one-by-one before being allowed to proceed. The upper limit $\lambda = 1$ could describe an accident in which one lane is blocked but the traffic authorities have not had time to impose a new speed limit in the remaining lane.

We will adopt the model in which the velocity of the vehicles depends only on the traffic density and use the velocity-density law

$$V_L^* = V_{L\max}^* \left(1 - \frac{\rho_L^*}{\rho_{L\max}^*}\right) \ , \quad V_R^* = V_{R\max}^* \left(1 - \frac{\rho_R^*}{\rho_{R\max}^*}\right) \ . \tag{2.4}$$

We chose $\rho_{L\max}^*$ as the characteristic traffic density, $V_{L\max}^*$ as the characteristic velocity and $\rho_{L\max}^*\, V_{L\max}^*$ as the characteristic flux of vehicles. The characteristic quantities apply in the road block as well as in the open road. We define the dimensionless variables

$$\begin{aligned} \rho_L &= \frac{\rho_L^*}{\rho_{L\,\max}^*}\ , \qquad V_L = \frac{V_L^*}{V_{L\,\max}^*}\ , \qquad q_L = \frac{q_L^*}{\rho_{L\,\max}^* V_{L\,\max}^*}\ , \\ \rho_R &= \frac{\rho_R^*}{\rho_{L\,\max}^*}\ , \qquad V_R = \frac{V_R^*}{V_{L\,\max}^*}\ , \qquad q_R = \frac{q_R^*}{\rho_{L\,\max}^* V_{L\,\max}^*}\ . \end{aligned} \tag{2.5}$$

Then from (2.4),

$$\begin{aligned} V_L &= 1 - \rho_L\ , \qquad q_L = \rho_L(1-\rho_L)\ , \\ V_R &= \lambda(1-2\rho_R)\ , \quad q_R = \lambda\rho_R(1-2\rho_R)\ . \end{aligned} \tag{2.6}$$

The road block is removed after a time T^*. The characteristic time is chosen to be T^* and the characteristic length is therefore $T^* V_{L\,\max}^*$. We define

$$t = \frac{t^*}{T^*}\ , \quad x = \frac{x^*}{T^* V_{L\,\max}^*}\ , \quad L = \frac{L^*}{T^* V_{L\,\max}^*}\ . \tag{2.7}$$

Unless otherwise stated, dimensionless variables will be used.

$$\rho_L^2 - \rho_L + \frac{\lambda}{8} = 0 \tag{3.2}$$

which has two roots,

$$\rho_A(\lambda) = \frac{1}{2}\left[1 - \left(1 - \frac{\lambda}{2}\right)^{1/2}\right] = \frac{\lambda}{8} + \mathrm{O}(\lambda^2) \qquad \text{as} \quad \lambda \to 0\ , \tag{3.3}$$

$$\rho_B(\lambda) = \frac{1}{2}\left[1 + \left(1 - \frac{\lambda}{2}\right)^{1/2}\right] = 1 - \frac{\lambda}{8} + \mathrm{O}(\lambda^2) \quad \text{as} \quad \lambda \to 0\ , \tag{3.4}$$

with the properties

$$\rho_A + \rho_B = 1\ , \quad \rho_A\rho_B = \frac{\lambda}{8}\ . \tag{3.5}$$

A shock will not form at the road block if $0 < \rho_L \le \rho_A$ and $\rho_B \le \rho_L \le 1$.

When $\rho_A < \rho_L < \rho_B$, a shock forms at the entrance to the road block and propagates backwards into the oncoming traffic. It is assumed that the traffic flow in the road block will be such as to allow the maximum possible flux in the road block so that $q_R = 1/4$. The density could either increase across the shock from ρ_L to ρ_B or decrease from ρ_L to ρ_A. It can be shown that if the density decreases across the shock then the shock is unstable while if it increases then the shock is stable [4]. Hence ρ_L increases to ρ_B and the traffic flux at the entrance to the road block is continuous.

The velocity of the shock is determined from the Rankine-Hugoniot condition [3, 4]. Let q_-, ρ_-, V_- be the flux, density and vehicle velocity on the left side of the shock and q_+, ρ_+, V_+ be the corresponding values on the right side of the shock. By considering continuity of the flux of vehicles relative to the shock it is found that the dimensionless shock velocity

$$U = \frac{[q]}{[\rho]} = \frac{q_+ - q_-}{\rho_+ - \rho_-} \tag{3.6}$$

or

$$\frac{dS}{dt} = U = 1 - \rho_- - \rho_+ \; . \tag{3.7}$$

where $x = S(t)$ is the position of the shock.

For the shock which forms at the entrance to the road block $x = S_L(t)$, $\rho_- = \rho_L$, $\rho_+ = \rho_B$ and $S_L(0) = 0$. We assume that ρ_L is constant. Then since $1 - \rho_B = \rho_A$,

$$S_L(t) = -(\rho_L - \rho_A)t \; . \tag{3.8}$$

Since $\rho_L > \rho_A$ the shock propagates backwards into the oncoming traffic. In dimensional units the length of the tailback at time t^* after the shock was first formed is

$$\left| S_L^*(t^*) \right| = (\rho_L - \rho_A)t^* V^*_{L\,\mathrm{max}} \; . \tag{3.9}$$

Consider now exit from the road block and the open road $x > L$. Because of continuity of traffic flux at the exit point $x = L$ the traffic flux in the open road beyond the road block is $\lambda/8$. From Figure 1 the traffic density in the road block, $\rho_B = 1/4$, could either decrease to ρ_A or increase to ρ_B in the open road. Because the road changes from one lane to two lanes at $x = L$ the traffic density will clearly decrease to ρ_A. Now the density of the traffic which passed through the position of the road block before it was in place is ρ_L. A shock discontinuity, $x = S_R(t)$, therefore propagates in the open road $x > L$ with $\rho_- = \rho_A$ and $\rho_+ = \rho_L$. The velocity of the shock from the Rankine-Hugoniot condition (3.7) is

$$\frac{dS_R}{dt} = 1 - \rho_A - \rho_L = \rho_B - \rho_L \tag{3.10}$$

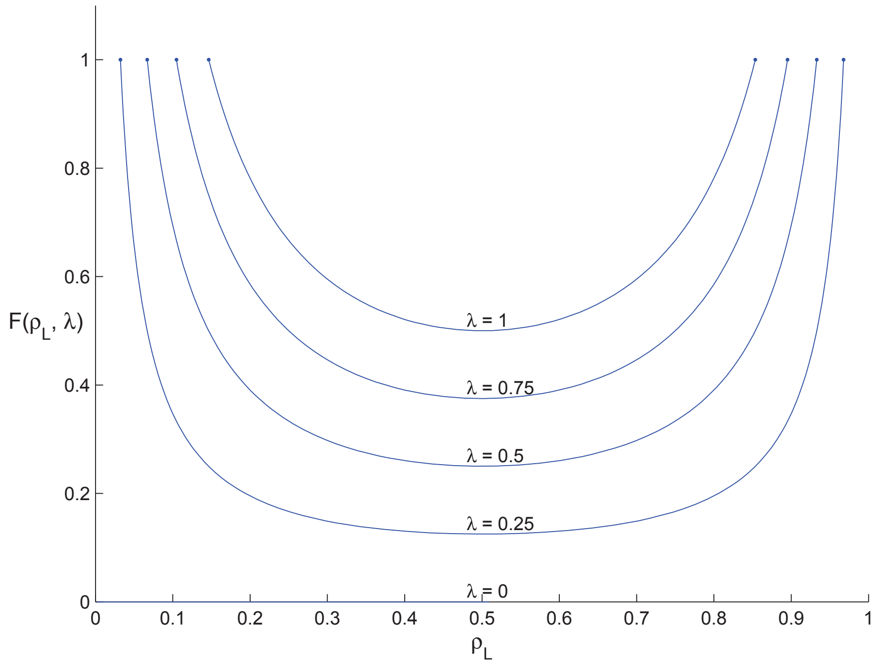

Figure 2: Ratio $F(\rho_L, \lambda)$ of the traffic flux out of the road block to the traffic flux into the left shock plotted against the traffic density on the open road ρ_L for $\rho_A \leq \rho_L \leq \rho_B$ and for a range of values of λ.

and since $S_R(0) = L$,

$$S_R(t) = L + (\rho_B - \rho_L)t \; . \tag{3.11}$$

Since $\rho_B > \rho_L$ the shock propagates in the positive x-direction. We see from Figure 1 that there is a decrease in the traffic flux from q_L to $\lambda/8$ due to the road block. Thus

$$F(\rho_L, \lambda) = \frac{\text{traffic flux out of road block}}{\text{traffic flux into left shock}} = \frac{\lambda}{8\rho_L(1-\rho_L)} \; , \tag{3.12}$$

where $\rho_A \leq \rho_L \leq \rho_B$. Since $\lambda/8 \leq q_L \leq 1/4$,

$$\frac{\lambda}{2} \leq F(\rho_L, \lambda) \leq 1 \; . \tag{3.13}$$

The minimum value of $F(\rho_L, \lambda)$ occurs for $\rho_L = 1/2$ and the maximum value when $\rho_L = \rho_A$ and $\rho_L = \rho_B$. Graphs of $F(\rho_L, \lambda)$ against ρ_L for $\rho_A \leq \rho_L \leq \rho_B$ and for a range of values of λ are plotted in Figure 2. Clearly the traffic flux is most adversely affected by the road block when the flux on the open road is close to the maximum value which is attained for $\rho_L = 1/2$. The adverse effect on the flux can be reduced by increasing the speed limit in the road block and therefore increasing λ.

The traffic density profile after the road block has been in place for time $t = 1$ is plotted in Figure 3. The density of oncoming traffic ρ_L is assumed constant throughout the time.

4 Removal of road block

The road block is removed at dimensionless time $t = 1$. The initial condition for the density distribution is given in Figure 3. We will investigate the maximum length to which the tailback will grow and the time it takes for the effects of the road block to dissipate.

The traffic density ρ and the traffic flux $q = Q(\rho)$ satisfy the conservation equation in dimensionless form

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0 \ . \tag{4.1}$$

Equation (4.1) can be written as

$$\frac{\partial \rho}{\partial t} + \frac{dQ}{d\rho}\,\frac{\partial \rho}{\partial x} = 0 \ . \tag{4.2}$$

Consider the traffic flow outside the road block. Then $Q(\rho) = \rho(1 - \rho)$ and (4.2) becomes

$$\frac{\partial \rho}{\partial t} + (1 - 2\rho)\,\frac{\partial \rho}{\partial x} = 0 \tag{4.3}$$

and this equation holds of all values of x when $t > 1$. The initial conditions are

$$\rho(x, 1) = f(x) \tag{4.4}$$

where $f(x)$ is defined by Figure 3.

Using the theory of characteristic for first order quasi-linear partial differential equations ([1] to [5] it can be verified that the characteristic projections

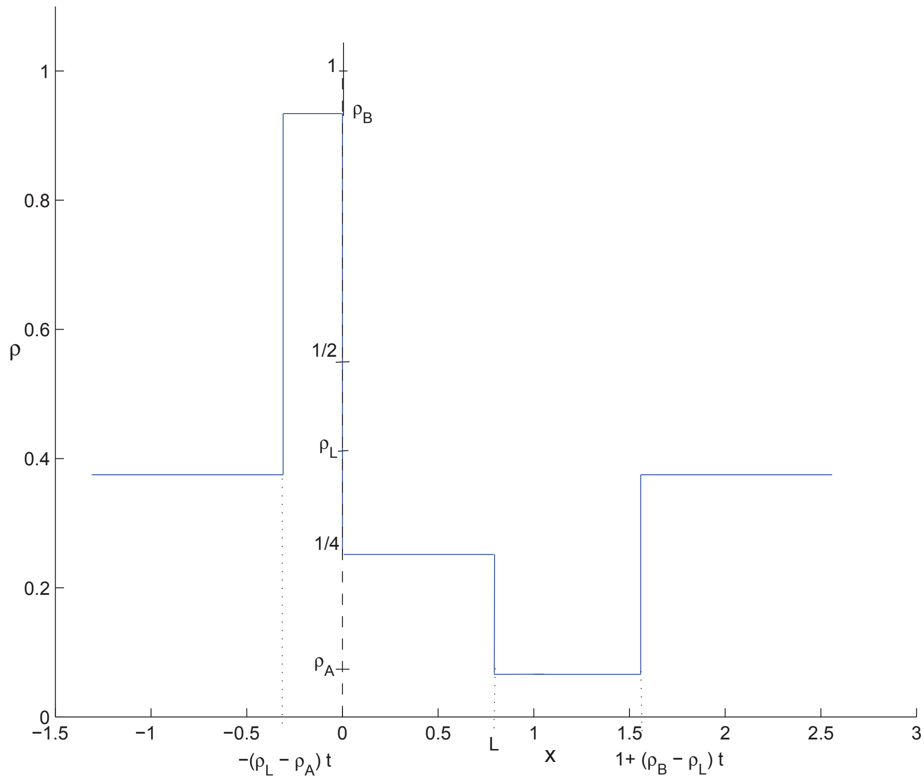

Figure 3: Traffic density profile after the road block has been in place for time t with $\rho_L = 3/8$, $\lambda = 1/2$ and t=1.

on the (x, t) plane are the one-parameter family of straight lines

$$t - 1 = \frac{x}{1 - 2f(\sigma)} - \frac{\sigma}{1 - 2f(\sigma)} \, , \qquad f(\sigma) \neq \frac{1}{2} \tag{4.5}$$

$$x = \sigma \, , \qquad f(\sigma) = \frac{1}{2} \tag{4.6}$$

and that on these lines

$$\rho(s, \sigma) = f(\sigma) \, , \tag{4.7}$$

where σ is the parameter along the initial curve and s is the parameter along the characteristic curves.

The critical traffic density for maximum flux is $\rho_L = 1/2$. Light traffic occurs when $0 < \rho_L < 1/2$ and heavy traffic when $1/2 < \rho_L \leq 1$. The evolution of the traffic when the road block is removed depends on whether the traffic density is light, critical or heavy.

4.1 Light traffic

We first relate to the characteristic projections the left shock which forms at the entrance to the road block. There are two families of characteristic projections. The equation of the characteristics emanating from $t = 0$, $x < 0$, is obtained by replacing $t - 1$ by t and $f(\sigma) = \rho_L$ in (4.5):

$$t = \frac{x}{1 - 2\rho_L} - \frac{\sigma}{1 - 2\rho_L} \, . \tag{4.8}$$

Along the characteristic lines (4.8), $\rho = \rho_L$. The characteristics emanating from the road block, $x = 0$, $t > 0$, are obtained from (4.5) by replacing $t - 1$ by t and $f(\sigma) = \rho_B$. Since $\rho_A + \rho_B = 1$ their equation can be written as

$$t = -\frac{x}{(\rho_B - \rho_A)} + \frac{\sigma}{(\rho_B - \rho_A)} \, . \tag{4.9}$$

The two families of characteristic projections intersect immediately and a shock forms at $x = 0$, $t = 0$. Using the Rankine-Hugonoit condition we found that the equation of this shock is (3.8) and is the straight line from the origin to the point P in Figure 4.

When the road block is removed at time $t = 1$, the density distribution is given by Figure 3. An expansion fan forms at the point $(0, 1)$ in the (x, t) plane. The equations of the characteristics in the fan are obtained from (4.5) and (4.6) by putting $\sigma = 0$ and $f(\sigma) = \rho$ where ρ takes all the values in the range $1/4 \leq \rho \leq \rho_B$:

$$t - 1 = \frac{x}{1 - 2\rho} \, , \qquad \frac{1}{4} \leq \rho < \frac{1}{2} \, , \qquad \frac{1}{2} < \rho \leq \rho_B \, , \tag{4.10}$$

$$x = 0 \, , \qquad \rho = \frac{1}{2} \, . \tag{4.11}$$

The limiting characteristics in the fan are, since $\rho_A + \rho_B = 1$,

$$\rho = \rho_B \; : \; t = 1 - \frac{x}{(\rho_B - \rho_A)} \, , \tag{4.12}$$

$$\rho = \frac{1}{4} \; : \; t = 1 + 2x \, . \tag{4.13}$$

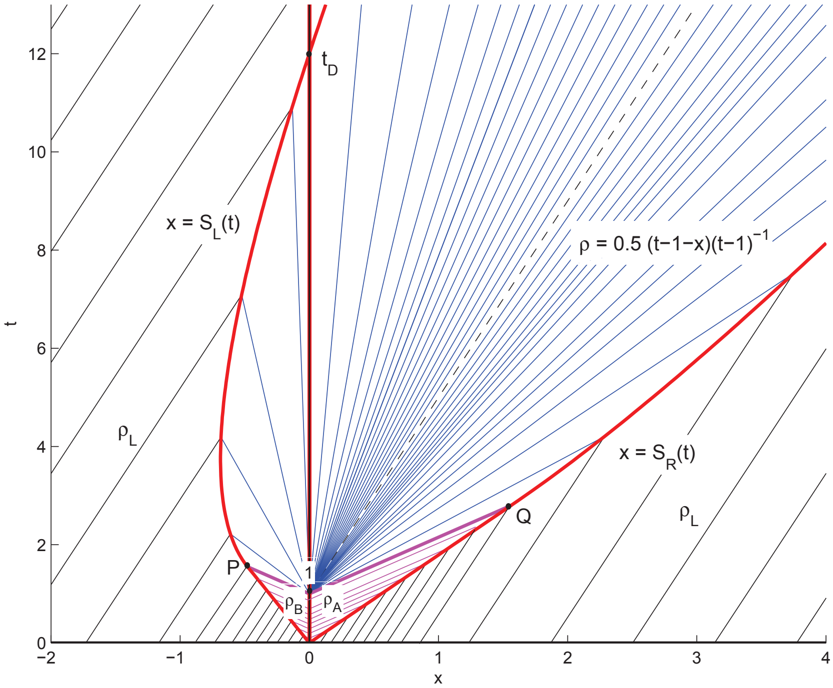

Figure 4: Light traffic. The characteristic projections and shock curves on the (x, t) plane for $\rho_L = 3/8$ and $\lambda = 1/2$. The left shock curve crosses the point $x = 0$ at $t_D = 12$. The dotted line starting at (0,1) is the dividing characteristic (4.34).

Solving (4.10) for ρ gives for the traffic density in the fan

$$\rho = \frac{t - 1 - x}{2(t - 1)}\ , \quad t > 1\ . \tag{4.14}$$

When $x = 0$, $\rho = 1/2$ for $t > 1$. The traffic flux is therefore a maximum for $t > 1$ at the point of entry to the road block when the road block is removed.

The limiting characteristic (4.12) meets the shock line (3.8) at the point P,

$$x_P = -\frac{(\rho_L - \rho_A)(\rho_B - \rho_A)}{(\rho_B - \rho_L)} \ , \quad t_P = \left(\frac{\rho_B - \rho_A}{\rho_B - \rho_L} \right) . \tag{4.15}$$

At $t = t_P$, $x = x_P$, the strength of the shock starts to decrease because the dissipation of the tailback has reached the shock discontinuity. The shock curve now separates the characteristic projections (4.8) and (4.10) with

$$\rho_- = \rho_L \ , \quad \rho_+ = \frac{t - 1 - S_L(t)}{2(t-1)} \ . \tag{4.16}$$

The Rankine-Hugoniot condition (3.7) yields the first order ordinary differential equation for the shock curve

$$\frac{dS_L}{dt} - \frac{1}{2(t-1)} \, S_L = \frac{1}{2} \, (1 - 2\rho_L) \ . \tag{4.17}$$

The solution of (4.17) subject to the initial condition $t = t_P$, $x = x_P$, is

$$S_L(t) = (1 - 2\rho_L)(t-1) - 2\Big[(\rho_L - \rho_A)(\rho_B - \rho_L)\Big]^{1/2} (t-1)^{1/2} \ . \tag{4.18}$$

The strength of the shock (4.18) is

$$\Delta\rho = \rho_+ - \rho_- = \left[\frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(t-1)} \right]^{1/2} \ , \tag{4.19}$$

which decreases steadily with time like $(t-1)^{-1/2}$. The shock curve (4.18) which starts at point P and the characteristic projections which intersect along the curve are shown in Figure 4.

For large values of $t - 1$,

$$S_L(t) \sim 2 \left(\frac{1}{2} - \rho_L \right) (t-1) \ , \tag{4.20}$$

which is positive for light traffic. The left shock curve therefore reverses its direction which occurs when

$$\frac{dS_L}{dt} = 0 \ . \tag{4.21}$$

At this time the shock velocity is instantaneously zero. Equation (4.21) is satisfied when

$$t - 1 = \frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)^2} \tag{4.22}$$

and occurs at the point

$$S_{L\min} = -\frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)} \,. \tag{4.23}$$

The maximum length of the tailback after the road block has been removed is, in dimensional units,

$$\mid S^*_{L\min} \mid = \frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)} \, T^* V^*_{L\max} \,. \tag{4.24}$$

The maximum length of the tailback is a linear function of T^*, the time the road block was in place and of $V^*_{L\max}$, the speed limit on the open road. The length of the tailback at time T^* when the road block is removed is given by (3.9) with $t^* = T^*$. Thus

$$D_L = \frac{\text{maximum length of tailback}}{\text{length of tailback when road block is removed}} = \frac{\rho_B - \rho_L}{1 - 2\rho_L} \,. \tag{4.25}$$

Graphs of $\mid S^*_{L\min} \mid$ scaled by $T^* V^*_{L\max}$ are plotted against ρ_L for $\rho_A \leq \rho_L < 1/2$ and for a range of values of λ in Figure 5. The maximum length of the tailback increases as λ decreases because of the decrease in the speed limit in the road block.

We will denote by t_D the time when the left shock again passes through the point $x = 0$ which was the position of the entrance to the road block when it was in place. At this time the traffic build-up to the left of the road block will have completely cleared. From (4.18), $S_L(t) = 0$ when

$$t_D - 1 = \frac{4(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)^2} \,. \tag{4.26}$$

The time difference (4.26) is four times the time taken after the road block has been removed for the tailback to reach its maximum length, given by (4.22). In Figure 6, the ratio of the time taken for the traffic build up to the left of the road block to clear, to the time that the road block was in place, is plotted against ρ_L for a range of values of λ. For $\rho_L = 3/8$ and $\lambda = 1$, 3/4, 1/2, 1/4 and 0, the ratio $t_D - 1 = 7$, 9, 11, 13, and 15. For instance, if $\rho_L = 3/8$ and $\lambda = 1/2$ and if the road block had been in place for $T^* = 0.5$ hr then the traffic build up would require an additional 5.5 hours to clear. Using the identities (3.5), equation (4.26) can be expressed in the alternative form

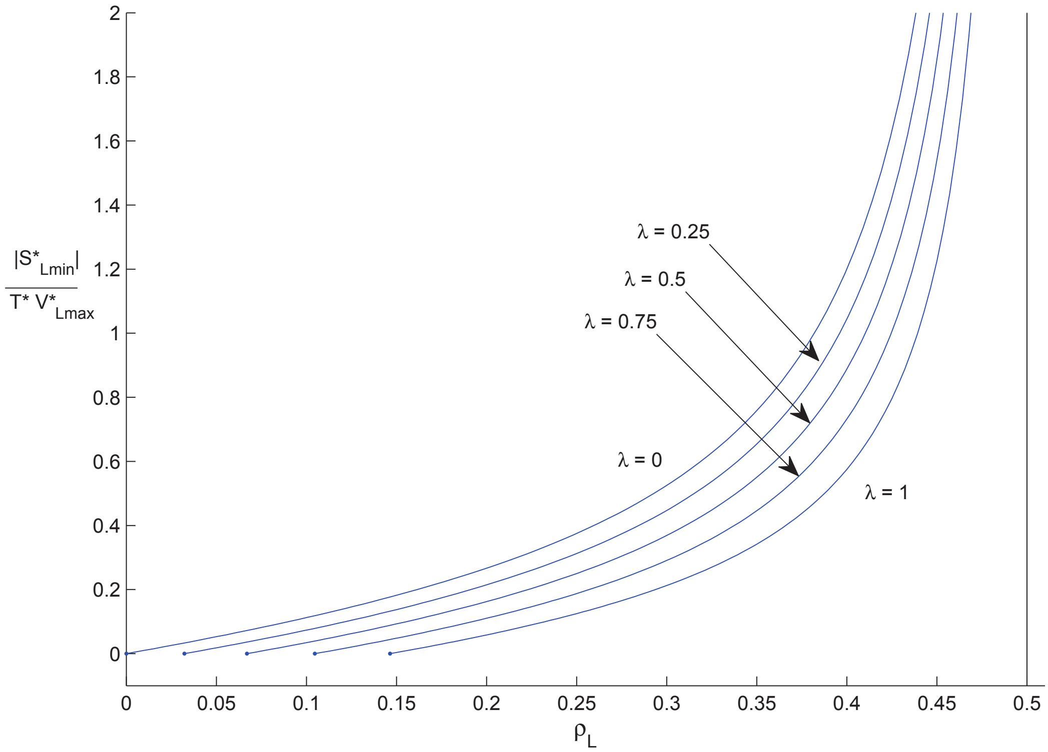

Figure 5: Road block removed. Maximum length of the tail-back, scaled by $T^*V^*_{L\,\max}$, plotted against the trafic density on the open road $\rho_A \le \rho_L < 1/2$ and for a range of values of λ.

$$t_D = \left(1 - \frac{\lambda}{2}\right) \frac{1}{(1-2\rho_L)^2} \,. \tag{4.27}$$

By putting $\lambda = 0$ in (4.27) the time at which the build up of traffic at a red traffic light clears is rederived [2]. At time t_D the traffic density at $x = 0$ suddenly decreases from the critical value $\rho = 1/2$ which existed for $1 \le t \le t_D$ to the value ρ_L, the traffic density before the road block was put in place. An observer at $x = 0$ would record the time t_D as the time that the traffic congestion caused by the road block finally cleared. For $1 \le t \le t_D$ this observer would have seen maximum traffic flux at $x = 0$.

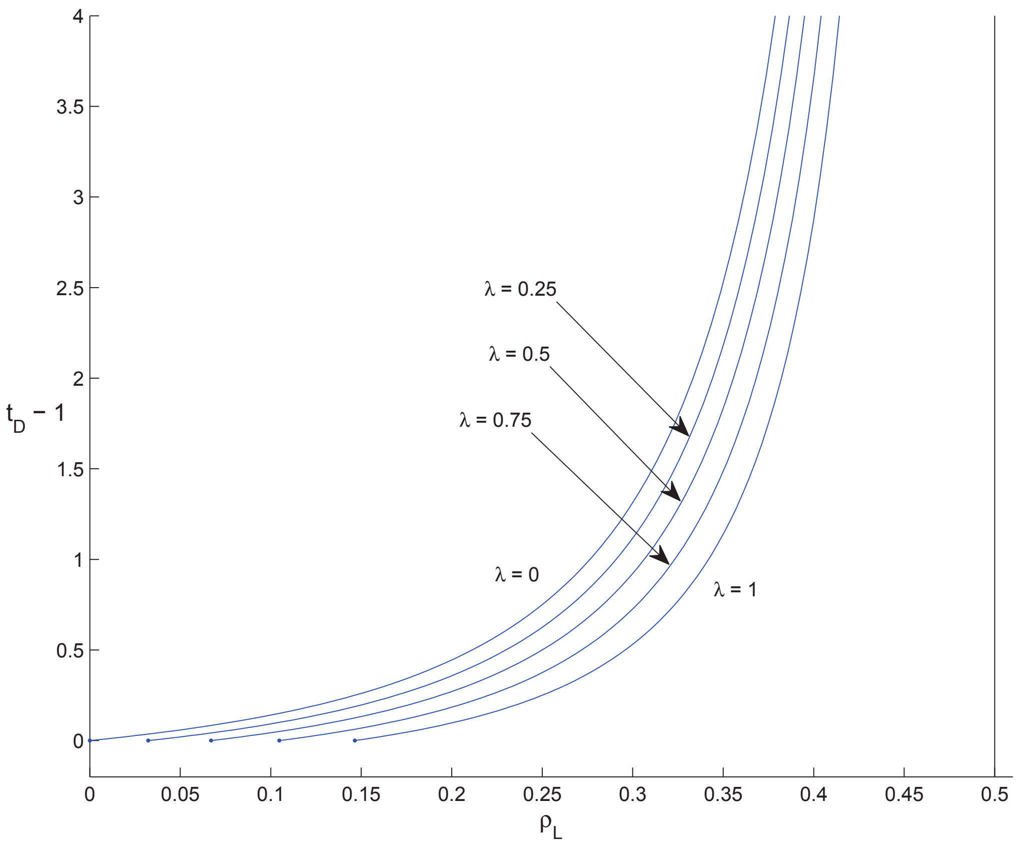

Figure 6: Ratio of the additional time taken for the effects of the road block to clear to the time the road block was in place, $t_D - 1$, plotted against ρ_L for $\rho_A \leq \rho_L < 1/2$ and for a range of values of λ.

Consider now the shock structure to the right of the road block. We will make the approximation that the time spent in the road block is much less than the time that the road block is in place, that is $L^* \ll T^* V^*_{R\max}$. The road block can then be approximated by the point $x = 0$, $t = 0$, in the (x, t) plane. The presence of the road block still determines the structure of the (x, t) plane through the traffic densities ρ_A and ρ_B.

Consider first the right shock, $x = S_R(t)$, which forms at the exit to the road block, $x = 0$. There are two families of characteristic projections. The equation of the characteristics emanating from $t = 0$, $x > 0$, are again given by (4.7). Along these characteristics, $\rho = \rho_L$. The characteristics emanating from the road block, $x = 0$, $t > 0$, are obtained from (4.5) by replacing $t - 1$ by t and $f(\sigma) = \rho_A$. Since $\rho_A + \rho_B = 1$, this family can be written as

$$t = \frac{x}{\rho_B - \rho_A} - \frac{\sigma}{\rho_B - \rho_A} \tag{4.28}$$

and compares with the family (4.9) for the left shock. Along the characteristics (4.28), $\rho = \rho_A$. The two families of characteristics, (4.8) and (4.28), intersect immediately and a shock forms at $x = 0$, $t = 0$. The equation of the shock curve is, from the Rankine-Hugoniot condition (3.7) and the initial condition $S_R(0) = 0$,

$$S_R(t) = (\rho_B - \rho_L)t \; . \tag{4.29}$$

Because the road block is approximated by a point at the origin, (4.29) replaces the shock equation (3.11). The shock (4.29) propagates in the positive x-direction and is represented by the straight line from the origin to Q in Figure 4.

Since the road block is approximated by a point the density in the expansion fan decreases from ρ_B directly to ρ_A instead of from ρ_B to 1/4 and from 1/4 to ρ_A. Instead of (4.13), the limiting characteristic in the fan is

$$t = 1 + \frac{x}{\rho_B - \rho_A} \; . \tag{4.30}$$

The limiting characteristic (4.30) meets the shock line (4.29) at the point Q,

$$x_Q = \frac{(\rho_B - \rho_L)(\rho_B - \rho_A)}{(\rho_L - \rho_A)} \; , \quad t_Q = \frac{(\rho_B - \rho_A)}{(\rho_L - \rho_A)} \; . \tag{4.31}$$

Since $0 < \rho_L < 1/2$, $t_Q > t_P$ and $x_Q >\mid x_P \mid$. At the point Q the strength of the right shock starts to decrease because the lead vehicle has caught up with the shock discontinuity. The right shock separates the families of characteristics (4.8) and (4.10) as for the left shock but instead of (4.16),

$$\rho_- = \frac{t - 1 - S_R(t)}{2(t-1)} \; , \quad \rho_+ = \rho_L \; . \tag{4.32}$$

The Rankine-Hugonoit condition again gives the differential equation (4.17) and the solution subject to the initial condition $t = t_Q$, $x = x_Q$ is

$$S_R(t) = (1 - 2\rho_L)(t-1) + 2\Big[(\rho_L - \rho_A)(\rho_B - \rho_L)\Big]^{1/2}(t-1)^{1/2} \; . \tag{4.33}$$

The limiting characteristic (4.12) meets the shock line (3.8) at the point P,

$$x_P = -\frac{(\rho_L - \rho_A)(\rho_B - \rho_A)}{(\rho_B - \rho_L)} \ , \quad t_P = \left(\frac{\rho_B - \rho_A}{\rho_B - \rho_L}\right) \ . \tag{4.15}$$

At $t = t_P$, $x = x_P$, the strength of the shock starts to decrease because the dissipation of the tailback has reached the shock discontinuity. The shock curve now separates the characteristic projections (4.8) and (4.10) with

$$\rho_- = \rho_L \ , \quad \rho_+ = \frac{t - 1 - S_L(t)}{2(t-1)} \ . \tag{4.16}$$

The Rankine-Hugoniot condition (3.7) yields the first order ordinary differential equation for the shock curve

$$\frac{dS_L}{dt} - \frac{1}{2(t-1)} \ S_L = \frac{1}{2} \ (1 - 2\rho_L) \ . \tag{4.17}$$

The solution of (4.17) subject to the initial condition $t = t_P$, $x = x_P$, is

$$S_L(t) = (1 - 2\rho_L)(t - 1) - 2\Big[(\rho_L - \rho_A)(\rho_B - \rho_L)\Big]^{1/2} (t-1)^{1/2} \ . \tag{4.18}$$

The strength of the shock (4.18) is

$$\Delta\rho = \rho_+ - \rho_- = \left[\frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(t-1)}\right]^{1/2} \ , \tag{4.19}$$

which decreases steadily with time like $(t-1)^{-1/2}$. The shock curve (4.18) which starts at point P and the characteristic projections which intersect along the curve are shown in Figure 4.

For large values of $t - 1$,

$$S_L(t) \sim 2\left(\frac{1}{2} - \rho_L\right)(t-1) \ , \tag{4.20}$$

which is positive for light traffic. The left shock curve therefore reverses its direction which occurs when

$$\frac{dS_L}{dt} = 0 \ . \tag{4.21}$$

At this time the shock velocity is instantaneously zero. Equation (4.21) is satisfied when

$$t - 1 = \frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)^2} \tag{4.22}$$

and occurs at the point

$$S_{L\,\min} = -\frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)} \ . \tag{4.23}$$

The maximum length of the tailback after the road block has been removed is, in dimensional units,

$$\mid S^*_{L\,\min} \mid = \frac{(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)}\, T^* V^*_{L\,\max} \ . \tag{4.24}$$

The maximum length of the tailback is a linear function of T^*, the time the road block was in place and of $V^*_{L\,\max}$, the speed limit on the open road. The length of the tailback at time T^* when the road block is removed is given by (3.9) with $t^* = T^*$. Thus

$$D_L = \frac{\text{maximum length of tailback}}{\text{length of tailback when road block is removed}} = \frac{\rho_B - \rho_L}{1 - 2\rho_L} \ . \tag{4.25}$$

Graphs of $\mid S^*_{L\,\min} \mid$ scaled by $T^* V^*_{L\,\max}$ are plotted against ρ_L for $\rho_A \le \rho_L < 1/2$ and for a range of values of λ in Figure 5. The maximum length of the tailback increases as λ decreases because of the decrease in the speed limit in the road block.

We will denote by t_D the time when the left shock again passes through the point $x = 0$ which was the position of the entrance to the road block when it was in place. At this time the traffic build-up to the left of the road block will have completely cleared. From (4.18), $S_L(t) = 0$ when

$$t_D - 1 = \frac{4(\rho_L - \rho_A)(\rho_B - \rho_L)}{(1 - 2\rho_L)^2} \ . \tag{4.26}$$

The time difference (4.26) is four times the time taken after the road block has been removed for the tailback to reach its maximum length, given by (4.22). In Figure 6, the ratio of the time taken for the traffic build up to the left of the road block to clear, to the time that the road block was in place, is plotted against ρ_L for a range of values of λ. For $\rho_L = 3/8$ and $\lambda = 1$, 3/4, 1/2, 1/4 and 0, the ratio $t_D - 1 = 7$, 9, 11, 13, and 15. For instance, if $\rho_L = 3/8$ and $\lambda = 1/2$ and if the road block had been in place for $T^* = 0.5$ hr then the traffic build up would require an additional 5.5 hours to clear. Using the identities (3.5), equation (4.26) can be expressed in the alternative form

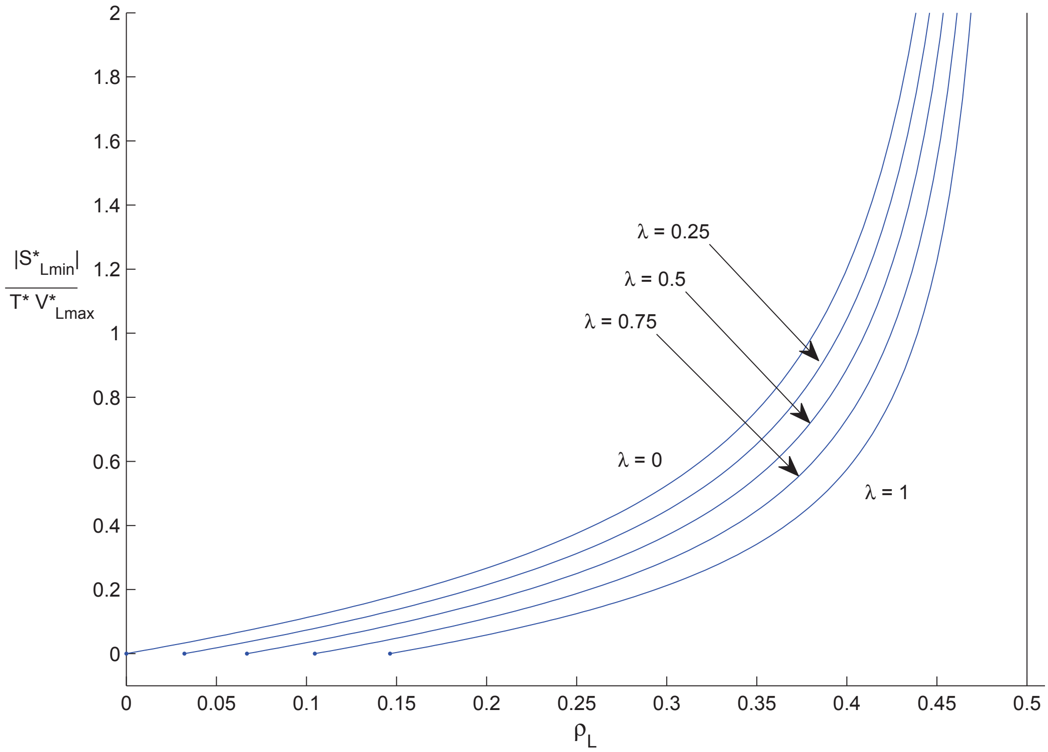

Figure 5: Road block removed. Maximum length of the tail-back, scaled by $T^*V^*_{L\max}$, plotted against the trafic density on the open road $\rho_A \leq \rho_L < 1/2$ and for a range of values of λ.

$$t_D = \left(1 - \frac{\lambda}{2}\right) \frac{1}{(1-2\rho_L)^2} \, . \tag{4.27}$$

By putting $\lambda = 0$ in (4.27) the time at which the build up of traffic at a red traffic light clears is rederived [2]. At time t_D the traffic density at $x = 0$ suddenly decreases from the critical value $\rho = 1/2$ which existed for $1 \leq t \leq t_D$ to the value ρ_L, the traffic density before the road block was put in place. An observer at $x = 0$ would record the time t_D as the time that the traffic congestion caused by the road block finally cleared. For $1 \leq t \leq t_D$ this observer would have seen maximum traffic flux at $x = 0$.

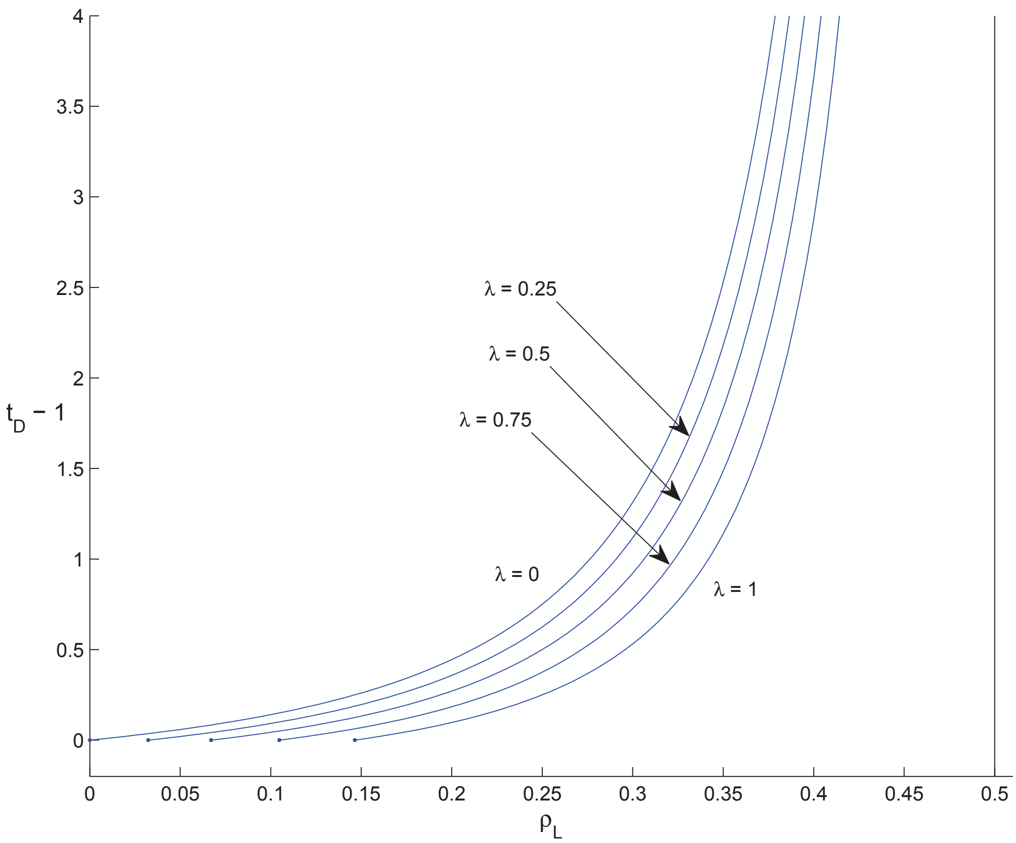

Figure 6: Ratio of the additional time taken for the effects of the road block to clear to the time the road block was in place, $t_D - 1$, plotted against ρ_L for $\rho_A \leq \rho_L < 1/2$ and for a range of values of λ.

Consider now the shock structure to the right of the road block. We will make the approximation that the time spent in the road block is much less than the time that the road block is in place, that is $L^* \ll T^* V^*_{R\max}$. The road block can then be approximated by the point $x = 0$, $t = 0$, in the (x, t) plane. The presence of the road block still determines the structure of the (x, t) plane through the traffic densities ρ_A and ρ_B.

Consider first the right shock, $x = S_R(t)$, which forms at the exit to the road block, $x = 0$. There are two families of characteristic projections. The equation of the characteristics emanating from $t = 0$, $x > 0$, are again given by (4.7). Along these characteristics, $\rho = \rho_L$. The characteristics emanating from the road block, $x = 0$, $t > 0$, are obtained from (4.5) by replacing $t - 1$ by t and $f(\sigma) = \rho_A$. Since $\rho_A + \rho_B = 1$, this family can be written as

$$t = \frac{x}{\rho_B - \rho_A} - \frac{\sigma}{\rho_B - \rho_A} \tag{4.28}$$

and compares with the family (4.9) for the left shock. Along the characteristics (4.28), $\rho = \rho_A$. The two families of characteristics, (4.8) and (4.28), intersect immediately and a shock forms at $x = 0$, $t = 0$. The equation of the shock curve is, from the Rankine-Hugoniot condition (3.7) and the initial condition $S_R(0) = 0$,

$$S_R(t) = (\rho_B - \rho_L)t \ . \tag{4.29}$$

Because the road block is approximated by a point at the origin, (4.29) replaces the shock equation (3.11). The shock (4.29) propagates in the positive x-direction and is represented by the straight line from the origin to Q in Figure 4.

Since the road block is approximated by a point the density in the expansion fan decreases from ρ_B directly to ρ_A instead of from ρ_B to 1/4 and from 1/4 to ρ_A. Instead of (4.13), the limiting characteristic in the fan is

$$t = 1 + \frac{x}{\rho_B - \rho_A} \ . \tag{4.30}$$

The limiting characteristic (4.30) meets the shock line (4.29) at the point Q,

$$x_Q = \frac{(\rho_B - \rho_L)(\rho_B - \rho_A)}{(\rho_L - \rho_A)} \ , \quad t_Q = \frac{(\rho_B - \rho_A)}{(\rho_L - \rho_A)} \ . \tag{4.31}$$

Since $0 < \rho_L < 1/2$, $t_Q > t_P$ and $x_Q > \mid x_P \mid$. At the point Q the strength of the right shock starts to decrease because the lead vehicle has caught up with the shock discontinuity. The right shock separates the families of characteristics (4.8) and (4.10) as for the left shock but instead of (4.16),

$$\rho_- = \frac{t - 1 - S_R(t)}{2(t-1)} \ , \quad \rho_+ = \rho_L \ . \tag{4.32}$$

The Rankine-Hugonoit condition again gives the differential equation (4.17) and the solution subject to the initial condition $t = t_Q$, $x = x_Q$ is

$$S_R(t) = (1 - 2\rho_L)(t-1) + 2\Big[(\rho_L - \rho_A)(\rho_B - \rho_L)\Big]^{1/2}(t-1)^{1/2} \ . \tag{4.33}$$

The strength of the shock is again given by (4.19). Since $1 - 2\rho_L > 0$ the shock does not reverse its direction but continues indefinitely in the positive x-direction as shown in Figure 4. The strength of the density discontinuity weakens with time like $(t-1)^{-1/2}$ but since the shock never returns to the position of the road block the shock discontinuity behind the road block never clears.

The characteristics in the expansion fan separate into two sets. The members of one set intersect the left shock and the members of the other set intersect the right shock. The equation of the dividing characteristic is

$$t = 1 + \frac{x}{1 - 2\rho_L} \tag{4.34}$$

which has the same gradient as the left and right shock curves as $t \to \infty$. It is parallel to the characteristic lines outside the region between the shock curves. Along the dividing characteristic, $\rho = \rho_L$ and the velocity of the traffic is the same as on the open road. The dividing characteristic is shown as the dotted line in Figure 4.

Finally consider the density profile at a time $t > 1$. The dissipation of the line of vehicles in front of the road block reaches the left shock at time t_P and the lead vehicle catches up with the right shock at time t_Q. Since $0 < \rho_L < 1/2$ for light traffic, $t_P < t_Q$ and the dissipation of the line of vehicles reaches the left shock before the lead vehicle catches up with the right shock. The traffic density at $x = 0$ jumps from $\rho = 1/2$ to $\rho = \rho_L$ at time t_D defined by (4.27). The density profile for light traffic after the road block has been removed is illustrated in Figure 7 for $\rho_L = 3/8$, $\lambda = 1/2$ and a range of values of time.

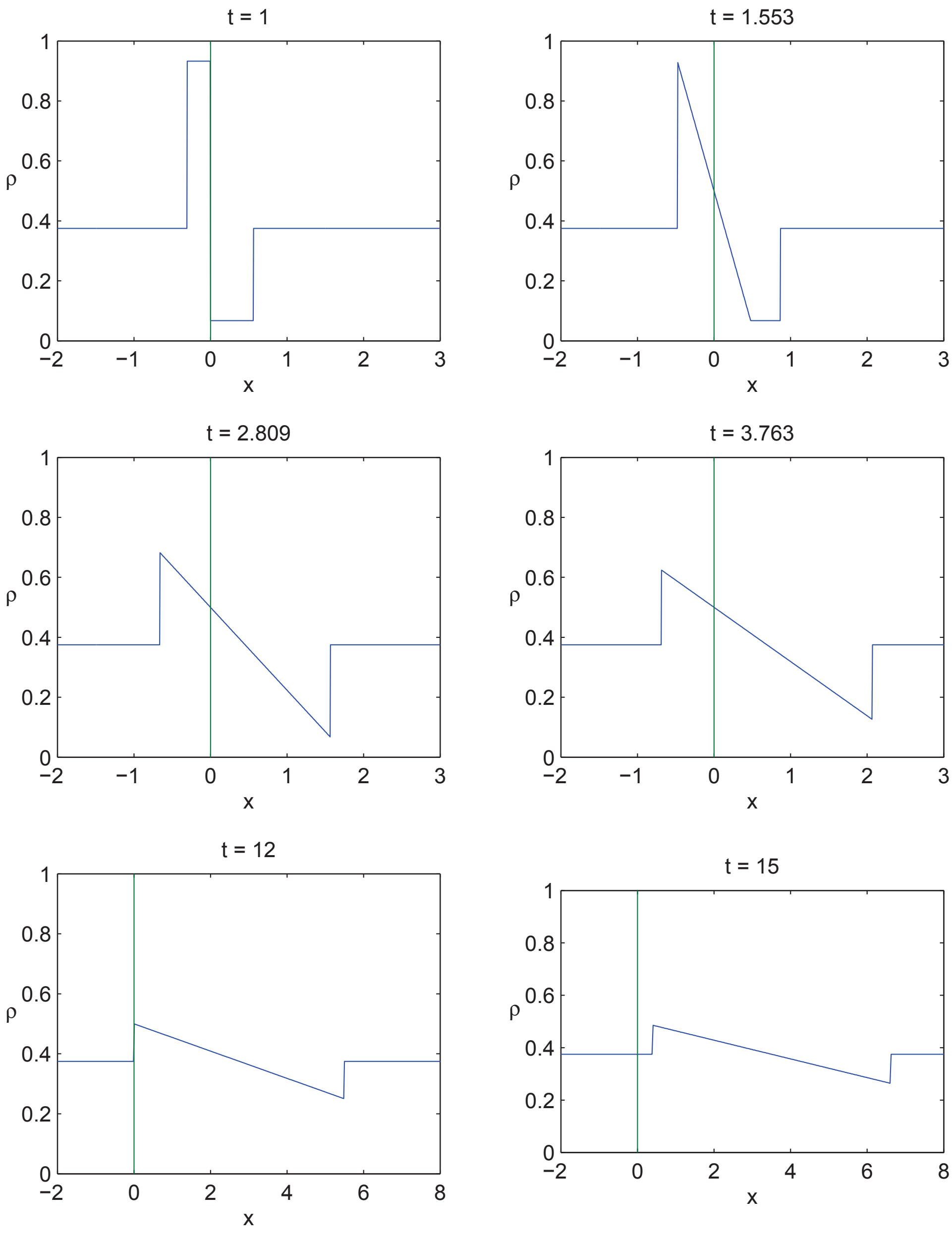

Figure 7: Light traffic. Traffic density plotted against x for $\rho_L = 3/8$, $\lambda = 1/2$ and $t = 1$, $t = t_P = 1.553$, $t = t_Q = 2.809$, $t = 3.763$, $t = t_D = 12$ and $t = 15$. The length of the tailback is maximum at $t = 3.763$.

4.2 Critical traffic flux, $\rho_L = 1/2$

When $\rho_L = 1/2$ the traffic flux is critical The characteristic projections starting from the line $t = 0$, $-\infty < x < \infty$, are now the vertical lines, $x = \sigma(-\infty < \sigma < \infty)$ and along the characteristics $\rho_L = 1/2$. The equations of the shocks which form at $x = 0$, $t = 0$ are

$$S_L(t) = -\frac{1}{2}(\rho_B - \rho_A)t \;, \quad S_R(t) = \frac{1}{2}(\rho_B - \rho_A)t \;. \tag{4.35}$$

The left shock changes at the point P,

$$x_P = -(\rho_B - \rho_A) \;, \quad t_P = 2 \;, \tag{4.36}$$

to the shock

$$S_L(t) = -(\rho_B - \rho_A)\Big[(t-1)\Big]^{1/2} \;, \tag{4.37}$$

while at

$$x_Q = +(\rho_B - \rho_A) \;, \quad t_Q = 2 \;, \tag{4.38}$$

the right shock changes to

$$S_R(t) = +(\rho_B - \rho_A)\Big[(t-1)\Big]^{1/2} \;. \tag{4.39}$$

The dividing characteristic in the fan is the vertical line $x = 0$, $t \geq 1$. The right shock travels forwards indefinitely while the left shock travels backwards indefinitely. The two shocks never return to the position where the road block was in place and the discontinuities in traffic density caused by the road block never disappear.

The characteristic projections and shock curves in the (x, t) plane and the density profiles for $\rho_L = 1/2$ after the road block has been removed are illustrated in Figures 8 and 9.

4.3 Heavy traffic

The equations of the characteristic projections and the shock curves are the same as for light traffic but now $1/2 < \rho_L < 1$. The diagram for the characteristic projections and shock curves for light traffic is reversed as shown in Figure 10. The results, which are exactly the same if we replace ρ_L by $1 - \rho_L$, x by $-x$ and $S_R(t)$ by $S_L(t)$ in the equations for light traffic, are shown in Figure 11. We see clearly the asymmetry between Figure 7 and Figure 11.

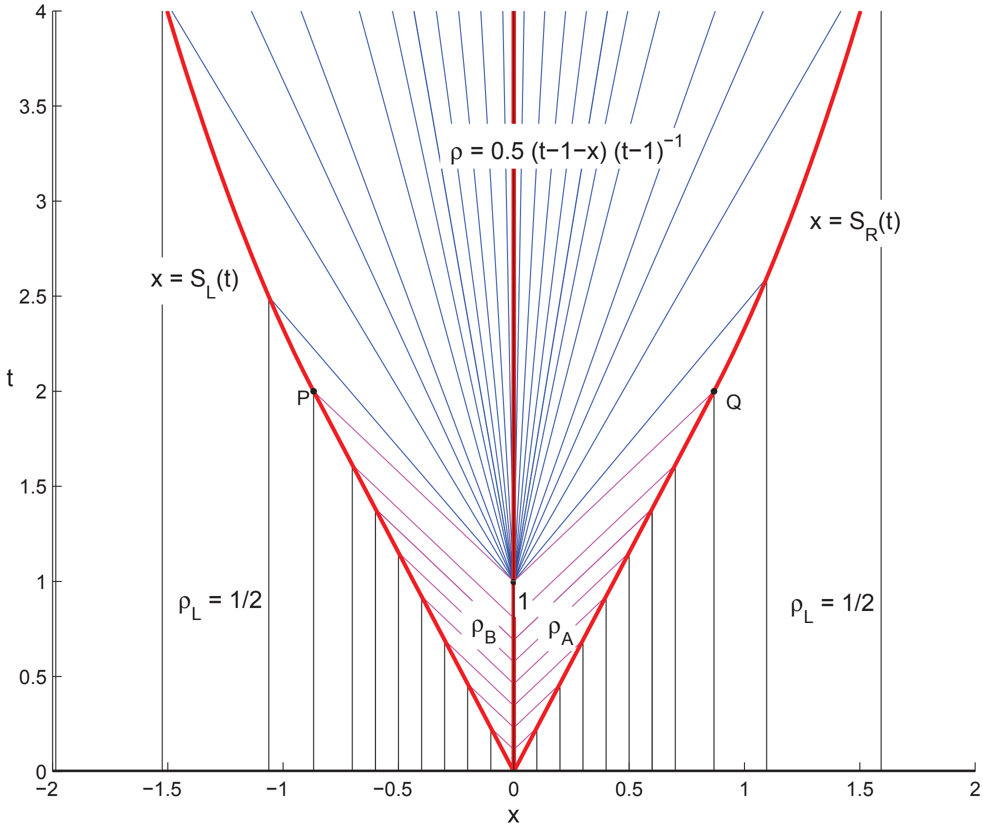

Figure 8: Critical traffic flux. The characteristic projections and shock curves in the (x, t) plane for $\rho_L = 1/2$ and $\lambda = 1/2$.

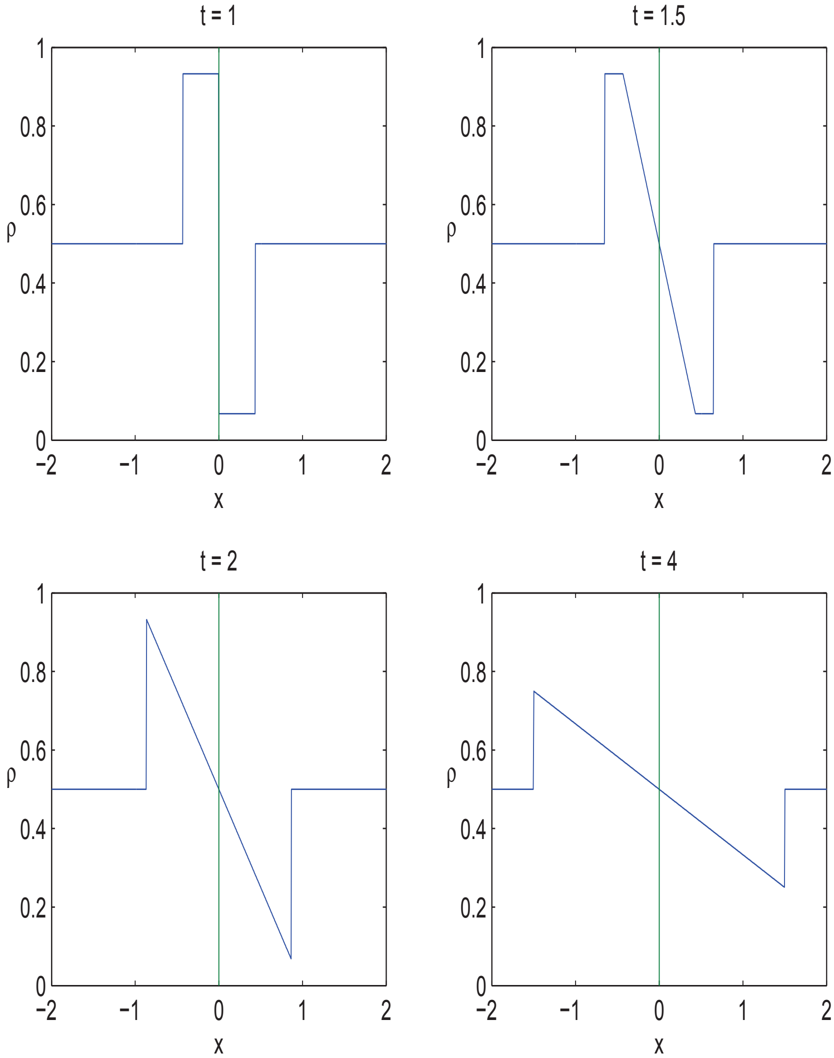

Figure 9: Critical traffic flux. Traffic density plotted against x for $\rho_L = 1/2$, $\lambda = 1/2$ and $t = 1$, $t = 1.5$, $t = t_P = t_Q = 2$ and $t = 4$.

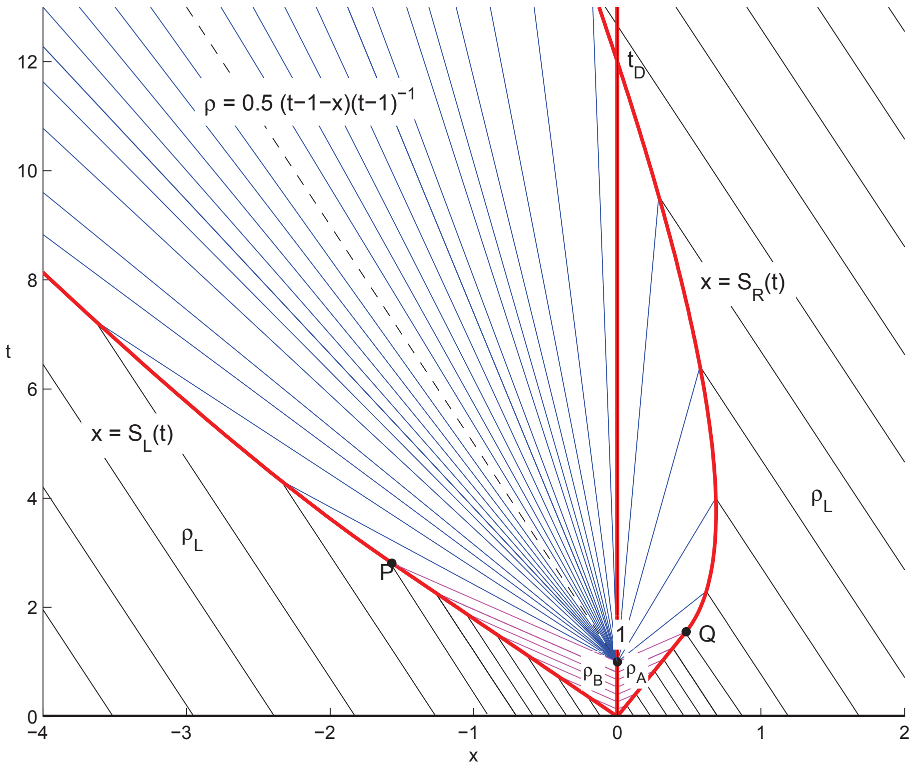

Figure 10: Heavy traffic. The characteristic projections and shock curves on the (x, t) plane for $\rho_L = 5/8$ and $\lambda = 1/2$. The right shock curve crosses the point $x = 0$ at $t_D = 12$. The dividing characteristic (4.34) has negative gradient and is the dotted line starting at (0,1).

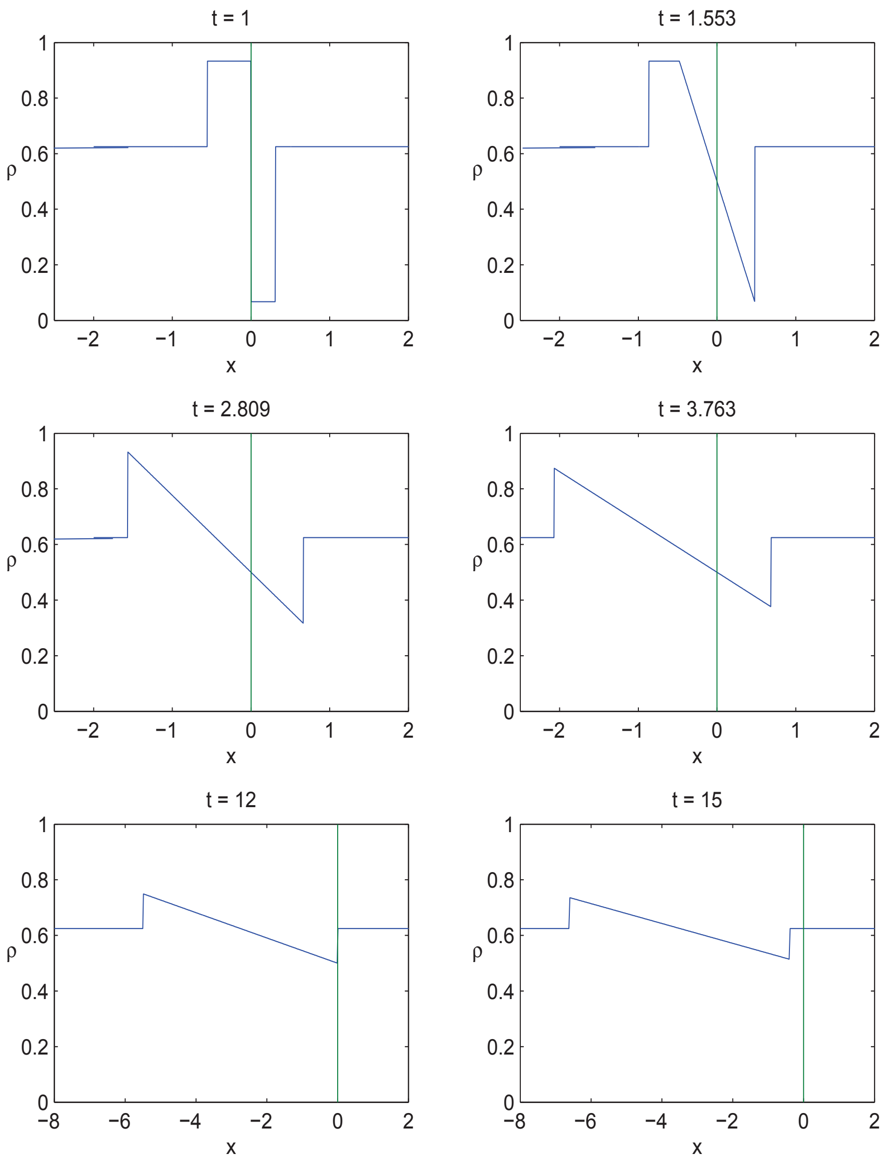

Figure 11: Heavy traffic. Traffic density plotted against x for $\rho_L = 5/8$, $\lambda = 1/2$ and $t = 1$, $t = t_Q = 1.553$, $t = t_P = 2.809$, $t = 3.763$. $t = t_D = 12$ and $t = 15$. At time $t = 3.763$ the shock has reached its maximum distance behind the road block.

5 Conclusions

In the model used in this paper, the effect of the road block on the traffic depended on three parameters, the density of the oncoming traffic ρ_L, the time T^* the road block was in place and the ratio λ of the speed limit in the road block to the speed limit in the open road.

The density ρ_L cannot be adjusted by the traffic authorities at the road block. We found from the graphs plotted against ρ_L that the effects of the road block on the traffic flow depend strongly on ρ_L and that there are significant differences between the flow of light traffic and heavy traffic. Usually the traffic on an open road will be 'light' and we have therefore considered this case in most detail. The parameters T^* and λ, however, can be adjusted by the traffic authorities at the road block. The length of the tailback and the time taken for the congestion caused by the road block to clear can be reduced by reducing T^* and increasing λ. When the traffic flux through the road block is maximised the traffic density in the road is $\rho_R = 1/4$ and the traffic velocity is one half the speed limit. The speed limit in the road block can therefore be set at twice the maximum practical speed for a vehicle to be waved down and stopped in the road block.

In the model considered here the traffic velocity depended linearly on the traffic density. It gave analytical results for all quantities and yielded qualitative insights into the effect of the road block on the traffic flow which should be helpful in managing the road block. More accurate quantitative predictions may require more sophisticated models in which the traffic velocity may depend on a power of the traffic density or on the density gradient [3, 5].

Acknowledgements

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